CHAPTER-4

CONIC SECTIONS

OBJECTIVE:

The conic sections (or conics) - the ellipse, the parabola and the hyperbola - play an important role both in mathematics and in the application of mathematics to engineering. The main objective of this chapter is, therefore, to deal with the construction of various types of plane curves such as ellipse, parabola and hyperbola, etc which are otherwise known as conic sections. These curves are very often used in engineering practices. Many engineering structures such as arches, bridges, dams, monuments, cooling towers, water channels, chimneys, roofs of stadiums, etc. involve geometries of conic sections. It is therefore very much necessary to study the nature of these curves together with some of their geometric properties and explain some of the convenient methods for construction of these curves.

CONIC SECTION:

Conics (conic sections) are essentially a class of curves which are obtained when a double cone is intersected by a plane at different angles relative to the axis of the double cone. There are three main types of conics: the ellipse, the parabola and the hyperbola. From the ellipse we obtain the circle as a special case, and from the hyperbola we obtain the rectangular hyperbola as a special case. The circle is a special case of the ellipse, and is of sufficient interest in its own right that it is sometimes called the fourth type of conic section.

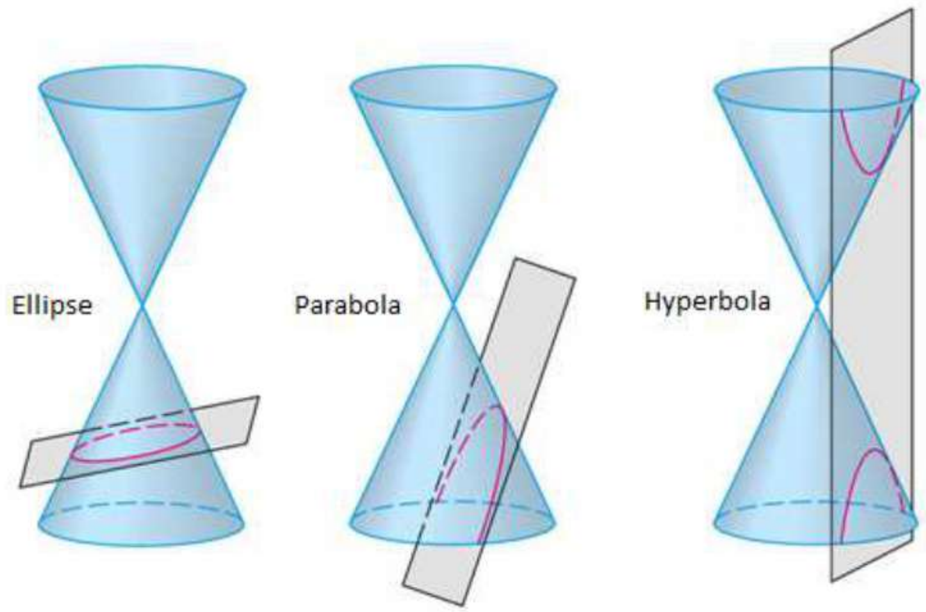


Figure 1

1. Ellipse

The section plane obtained by the intersection of a cutting plane, inclined to the axis of the cone and cutting all the generators, is called an ellipse. The angle of inclination of the cutting plane with the axis of the cone is more than that of the generator with the axis. An ellipse is a closed curve.

An imaginary line joining the apex to the centre of the base of the cone is known as the axis of the cone. The top point of the cone is called the apex.

2. Parabola

The section plane obtained by the intersection of a cutting plane, inclined to the axis of the cone and parallel to one of the generators, is called a parabola.

3. Hyperbola

The section plane obtained by the intersection of a cutting plane, inclined to the axis of the cone at an angle less than the inclination of the generator with the axis (semi-vertical angle, is called a hyperbola. In this case the cutting plane cuts the cone on one side of the axis.

4. Circle

The section plane obtained by the intersection of a cutting plane, parallel to the base of the cone, is called a circle. Circle is a special is the special case of the ellipse and is sometimes referred to as fourth type of conic section.

In analytic geometry, a conic may be defined as a plane algebraic curve of degree 2. There are a number of other geometric definitions possible. To have one such geometric definition, one has to know the term called *locus*.

Locus

The path traced out by a point when it moves in the space, under given conditions or in accordance with a definite law, is known as a locus of that point (loci in the plural).

Geometrical definition of conic/ conic section as a plane of *loci*

A conic section or conic is defined as the locus of a point moving in a plane in such a way that the ratio of its distances from a fixed point and a fixed straight line is always constant. The fixed point is called the *focus* and the fixed line is called the *directrix*.

The ratio $\frac{disance\ of\ the\ point\ from\ the\ focus}{distance\ of\ the\ point\ from\ the\ directrix}$ is called eccentricity and is denoted by *e*.

For ellipse, this eccentricity is always less than 1 (e < 1). It is equal to 1 (e = 1) for parabola and greater than 1 (e > 1) for hyperbola.

The line passing through the focus and perpendicular to the directrix is called the axis. The point at which the conic cuts its axis is called the *vertex*.

Ellipse

Ellipse is the locus of a point moving in a plane in such a way that ratio of its distances from a fixed point (focus) and a fixed straight line (directrix) is constant and is always less than one. Ellipse is a closed curve having two foci and two directices.

Parabola

Parabola is locus of a point moving in a plane in such a way that the ratio of its distances from a fixed point and a fixed straight line is constant and is equal to one. Parabola is an open curve of conic section.

Hyperbola

Hyperbola is locus of a point moving in a plane in such a way that the ratio of its distances from a fixed point and a fixed straight line is constant and is always greater than one. Hyperbola is an open curve of conic section.

PROPERTIES OF CONIC SECTIONS

<u>*Properties of ellipse*</u>

The equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where a = half the major axis. B = half the minor axis. The origin is the centre of the ellipse. The vertices are at a distance a from the centre C on both sides of the x axis. For different values of x like 0, ±1, ± 2 etc. The corresponding values of $y = \pm b\sqrt{1}, \pm b\sqrt{1-\frac{1}{a^2}}, \pm b\sqrt{1-\frac{4}{a^2}}$ ....

For any particular value of x, there are two equal and opposite values of y. The curve therefore is symmetric about the x-axis. For every +ve or –ve value of x, the values of y are identical and hence the curve is symmetrical about the y-axix.

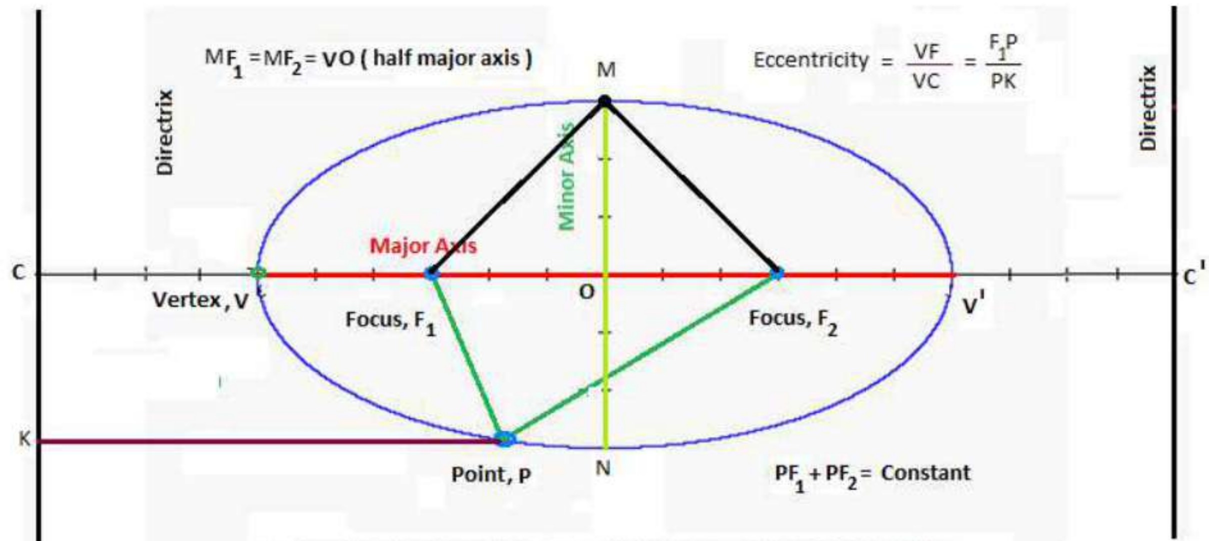
Properties of ellipse

1. Ellipse is the locus whose sum of distances from two fixed points is constant.

2. The length of the line segment from the end of a minor axis to a focus is the same as half the length of a major axis.

3. A light ray originating from one focus will pass through the opposite focus after reflecting off of the ellipse.

4. At all points on the ellipse, the sum of distances from the foci is 2a (twice the semi-major axis. This is another equation for the ellipse.



Diagramatical explanation of the mathematical properties of ellipse

From each point on the curve, the distance to the focus equals the distance to the "directrix

Every ray coming straight down is reflected to the focus

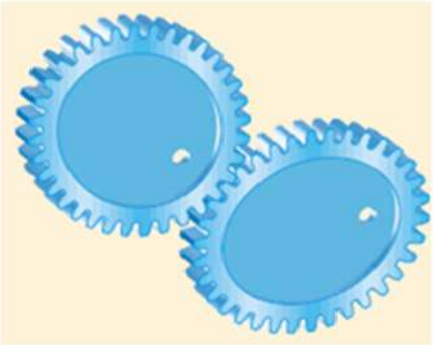
Applications of ellipse

There are many incident and uses of elliptical forms: orbits of satellites, planets, and comets; shapes of galaxies; gears and cams; some airplane wings, boat keels, and rudders; tabletops; public fountains; and domes in buildings are a few examples (see Fig. 2).

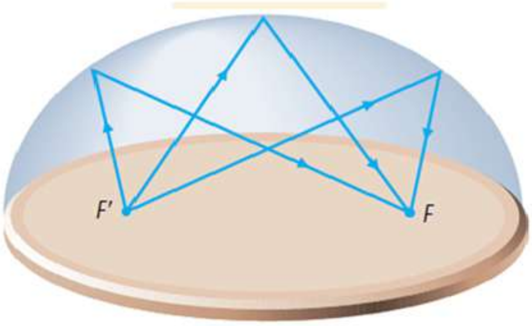
Figure 2(a) shows a pair of elliptical gears with pivot points at foci. Such gears transfer constant rotational speed to variable rotational speed, and vice versa. Figure 2(b) shows an elliptical dome. An interesting property of such a dome is that a sound or light source at one focus will reflect off the dome and pass through the other focus. The "whispering gallery" of the United States Senate is an ellipse. If you stand atone focus and whisper (speak quietly), you can be heard at the other focus (and nowhere else).Your voice is reflected off the walls to the other focus-following the path of the string.

Structural components with elliptical configuration have wider application in aerospace engineering and naval architecture. Figure 2(c) shows an aeroplane with its trailing edge, and wings and tails of elliptical shape. Of all possible wing shapes, it has been determined that the one with the least drag along the trailing edge is an ellipse. Use of elliptical reflectors and ultrasound to break up kidney stones is a fairly recent application in medicine. Shown in Figure 2(d) is a device called lithotripter which is used to generate intense sound waves that break up the stone from outside the body, thus avoiding surgery. The reflecting property of the ellipse is used to design and correctly position the lithotripter to ensure that the waves do not damage other parts of the body.

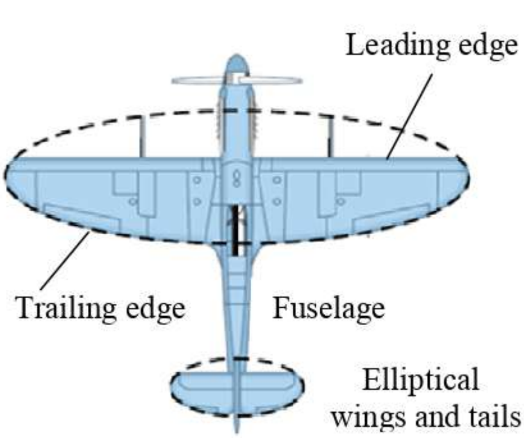
High-performance racing sailboats, shown in Figure 3, use elliptical keels, rudders, and main sails for the same reason as in the case of aeroplane wings.



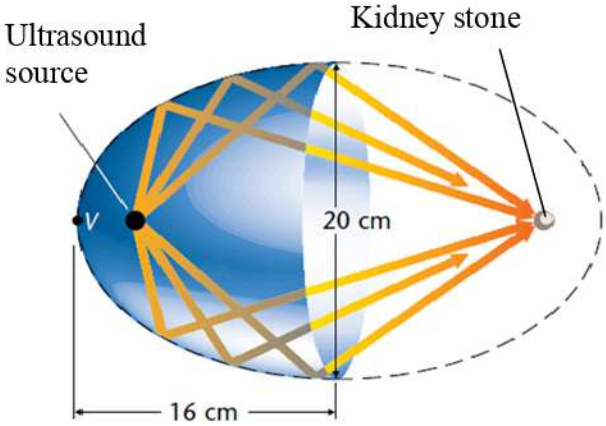
(a) Elliptical



(b) Elliptical

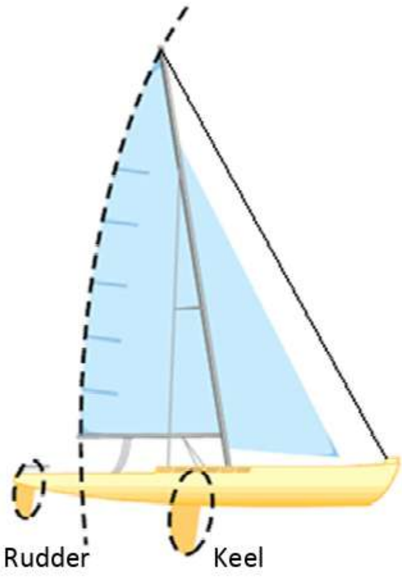


(c) Wing and trailing edge

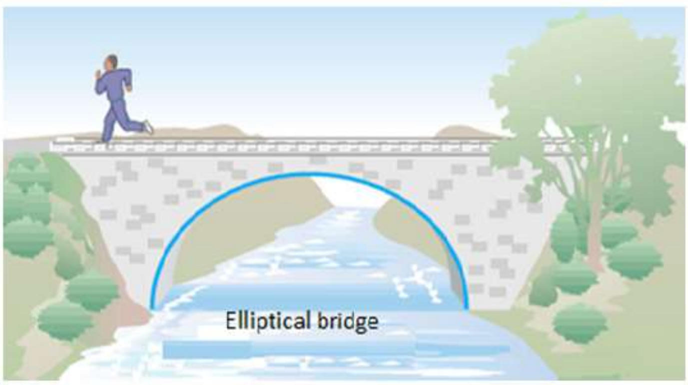


(d) Lithotripter

Figure 2



(a) Racing sail-boats



(b) Elliptical bridge

Figure 3

Methods of construction of ellipse

There are a number of methods available for construction of ellipse. The type of method to be used for construction of ellipse depends on the specific parameters of an ellipse. Some of the methods are mentioned as follows.

1. Eccentricity method
2. Concentric circle method
3. Arc of the circle method
4. Loop of thread method
5. Oblong method/Rectangle method
6. Trammel method
7. Four centres approximate method
8. Parallelogram method
9. Circumscribing parallelogram method

Of the above mentioned methods, the first three methods are included in the syllabus and need special attention so far as their constructional procedure is concerned.

1. Eccentricity method

This method is used when the eccentricity and the distance of the focus from the directrix are given. In the present case let us assume the eccentricity, e = 2/3

1. Draw a vertical line DD' as the directrix.
2. Draw CC' as the axis of the ellipse from any suitable point C on the directrix such that CC' is perpendicular to the directrix.
3. Mark a focus F on the axis such that CF is equal to the distance between the directrix and the focus.
4. Divide CF into 5 equal parts and mark the vertex on the 3rd division point from C. Thus $eccentricity = \frac{VF}{VC} = 2/3$
5. Draw a perpendicular VE equal to VF. Now draw a line joining C and E and prolong it in the direction of CE. Thus , in triangle, CVE, $\frac{VE}{VC} = \frac{VF}{VC} = \frac{2}{3}$ (since $VE = VF$)
6. Mark a point 1 on the axis and draw a perpendicular through it so as to meet prolonged CE at $1'$.
7. With F as centre and radius equal to $1-1'$, draw arcs to intersect the perpendicular through 1 at P_1 and P_1'.

The points P_1 and P_1' lie on the required ellipse because the ratio P_1 from DD' is equal to C1, $P_1F = 1 - 1'$ and $\frac{1-1'}{C1} = \frac{VF}{VC} = \frac{2}{3}$. Similarly, mark points 2, 3 etc. on the axis and obtain points P_2 and P_2', P_3 and P_3' etc.

8. Draw the ellipse through these points. The ellipse, so obtained is a closed curve with two foci and two directrices.

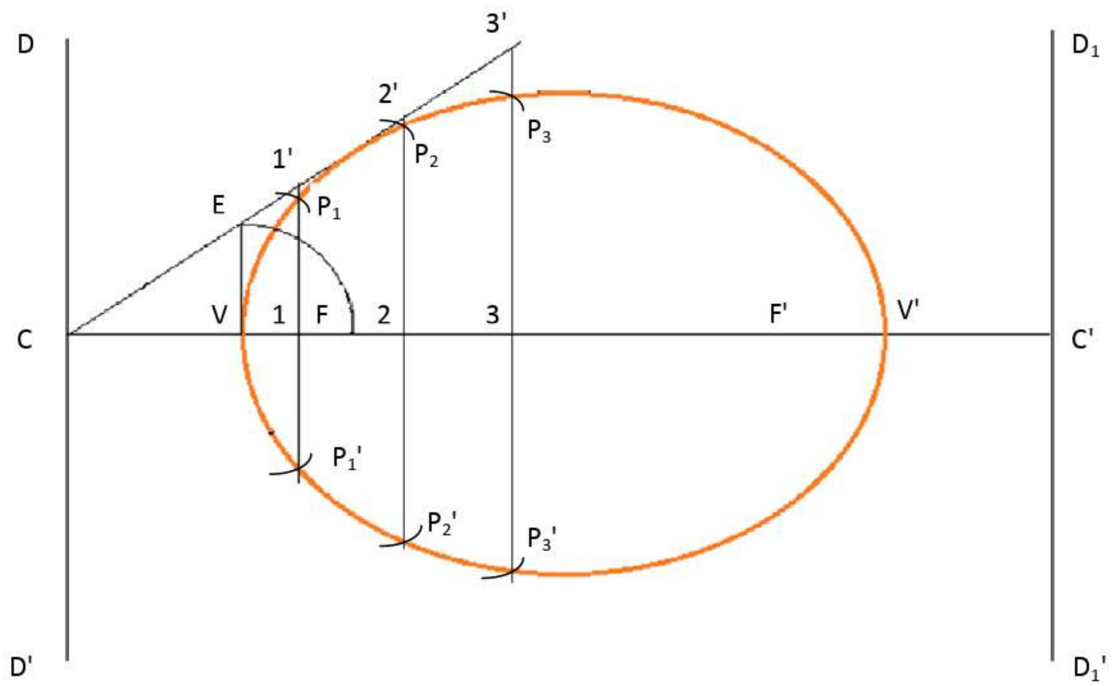


Figure 4

2. Concentric circle method

This method is used when the lengths of major and minor axes of an ellipse are given.

Procedure of construction

a. Draw the major axis and minor axes as AB and CD respectively so as to intersect each other at right angles at O as shown in the figure.
b. With centre O and AB and CD as diameters draw two circles.
c. Divide the circle described on the major axis (major-axis-circle) into a number of equal divisions, say 12 and mark 1,2 etc as shown.

d. Draw lines joining these points with the centre O and cutting the circle described on the minor axis (manor-axis-circle) at points $1', 2'$ etc.

e. Through point 1 on the major-axis-circle, draw a line parallel to CD, the minor axis.

f. Through point $1'$ on the minor-axis-circle, draw a line parallel to AB, the major axis.

g. Mark the point of intersection of the above two parallel lines as P_1. The point P_1, thus obtained, is a point on the required ellipse.

h. Repeat the construction through all the points and get more points.

i. Draw a smooth curve through all these points. The curve, so obtained, is the required ellipse.

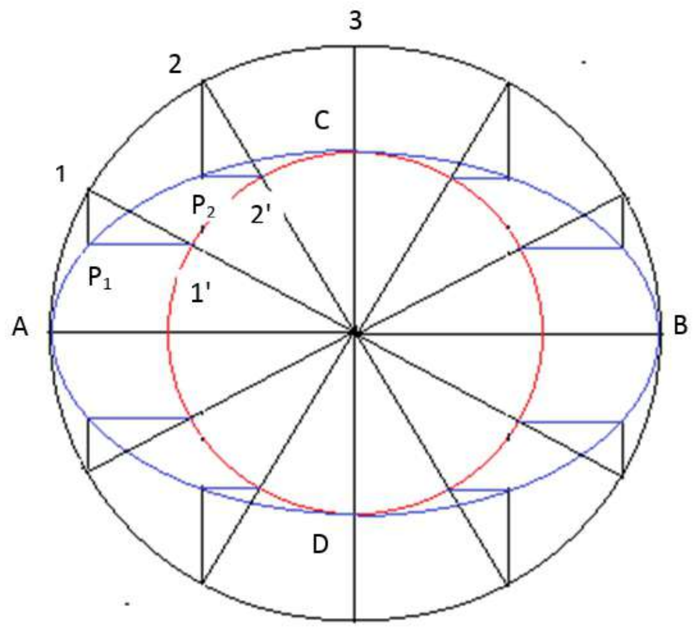


Figure 5

3. Arc of circle method

This method is also used when the lengths of major and minor axes of an ellipse are given.

a. Draw the major axis and minor axes as AB and CD respectively so as to intersect each other at right angles at O as shown in the figure.

b. Draw two arcs with C as centre and radius equal to AO (semi-major axis = half of AB) so as to cut major axis AB at F_1 and F_2. F_1 and F_2 are the foci of the ellipse.

c. Mark a number of points 1, 2, 3 etc. on AB.

d. With F_1 and F_2 as centres and A1 as radius, draw arcs on both sides of AB.

e. With same centres and radius equal to B1, draw arcs intersecting the previous arcs at four points marked as P_1.

f. Similar points are obtained with radii A2 and B2; and A3 and B3 etc.

g. Draw a smooth curve through all these points. The curve, so obtained, is the required ellipse.

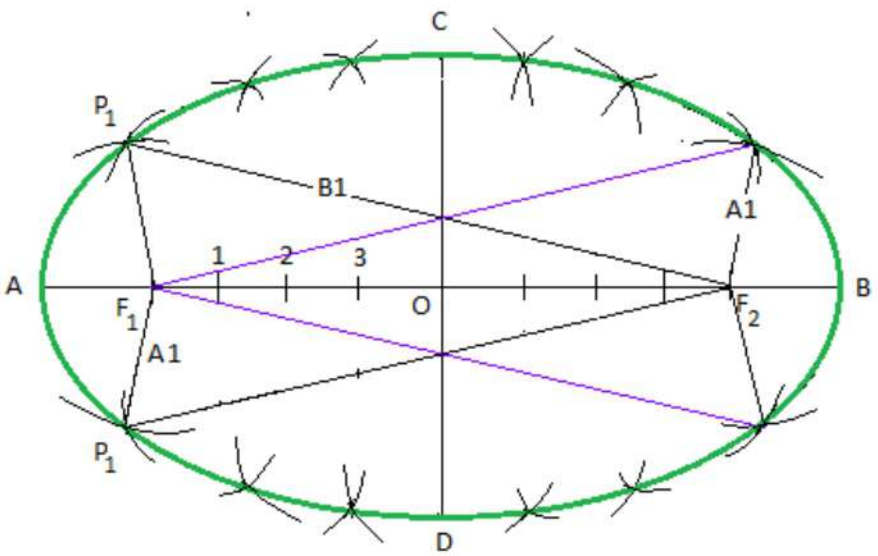


Figure 6

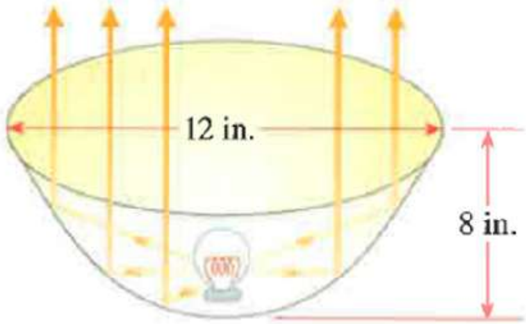
Properties of parabola

Remind students that a defining property for a parabola is the set of points P satisfying FP = PP'.

Application of parabola

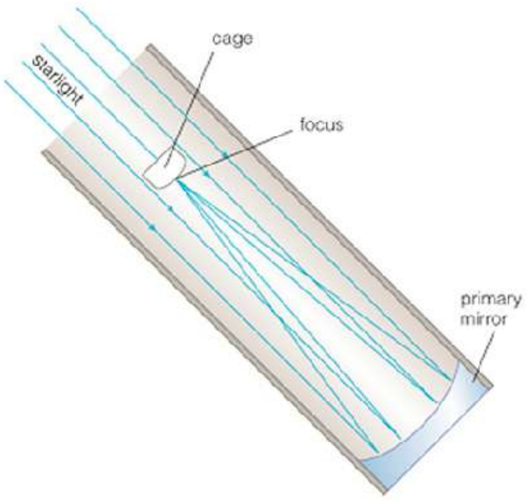
Parabolas have an important property that makes them useful as reflectors for lamps and telescopes. Light from a source placed at the focus of a surface of parabolic cross section will be reflected in such a way that it travels parallel to the axis of the parabola. Thus a parabolic mirror reflects light into a beam of parallel rays. Conversely, light approaching the reflector in rays parallel to its axis of symmetry is concentrated to the focus. This reflection property is used in the construction of reflecting telescopes.

In parabolic microphones, a parabolic reflector that reflects sound, but not necessarily electromagnetic radiation, is used to focus sound onto a microphone, giving it highly directional performance.
Unlike an inelastic chain, a freely hanging spring of zero unstressed length takes the shape of a parabola. Suspension-bridge cables are, ideally, purely in tension, without having to carry other, e.g. bending, forces. Similarly, the structures of parabolic arches are purely in compression.



(a) Parabolic reflector

(b) Satellite dish



(c) Telescope

(d) Solar cooker with parabolic reflector

Figure 7

(a) An array of parabolic troughs to collect solar energy

(b) A parabolic antenna

(c) A parabolic arch bridge (The parabolic arch is in compression)

Figure 8

Methods of construction of parabola

There are a number of methods available for construction of parabola. The type of method to be used for construction of parabola depends on the specific parameters of a parabola. Some of the methods are mentioned as follows.

1. Eccentricity method
2. Rectangle method
3. Tangent method
4. Measured abscissa method
5. Parallelogram method

Of the above mentioned methods, the first three methods are included in the syllabus and need special attention so far as their constructional procedure is concerned.

1. Eccentricity method

This method is used when the distance between the directrix and the focus is given.

a. Draw a vertical line DD' as the directrix.

b. Draw CC' as the axis of the parabola from any suitable point C on the directrix such that CC' is perpendicular to the directrix.

c. Mark focus F on the axis CC'.

d. Bisect CF to get the vertex V. $\left(since,\ eccentricity = \frac{VF}{VC} = 1 \right)$

e. Mark a number of points 1, 2, 3 etc. on the axis and through these points draw perpendiculars to the axis.

f. With F as centre and C1, C2, C3 etc. as radii, draw arcs so as to cut the perpendiculars through relevant points (perpendiculars through 1, 2, 3 etc.) at P_1 and P_1', P_1 and P_2', and P_1 and P_3' etc.

g. Draw a smooth curve through these points. The curve thus obtained is the required parabola.

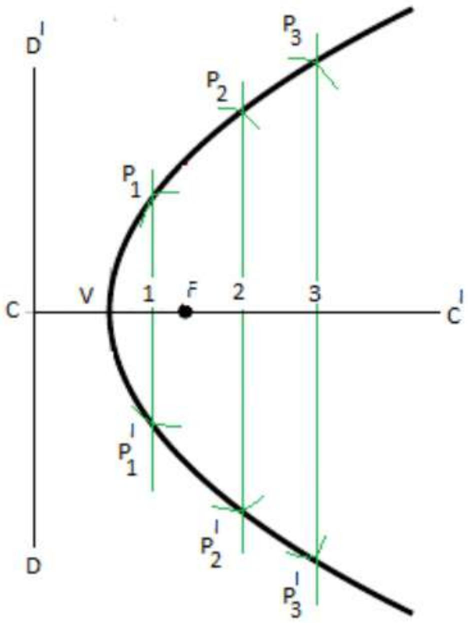


Figure 8

2. Rectangle method

This method is used when the base and the axis are given.

a. Draw a line AB as the base and mark the midpoint M on it.

b. Through M, draw the axis MN at right angles to AB.

c. Draw a rectangle ABCD with side BC equal to MN.

d. Divide AM and AD into same number of equal parts and name them as shown .Draw lines joining N with 1, 2 and 3.

e. Through 1', 2' and 3', draw perpendiculars to AB so as to intersect F1, F2 and F3 at points P_1, P_2 and P_3 respectively.

f. Draw a smooth curve through these points. The curve thus obtained is the required parabola.

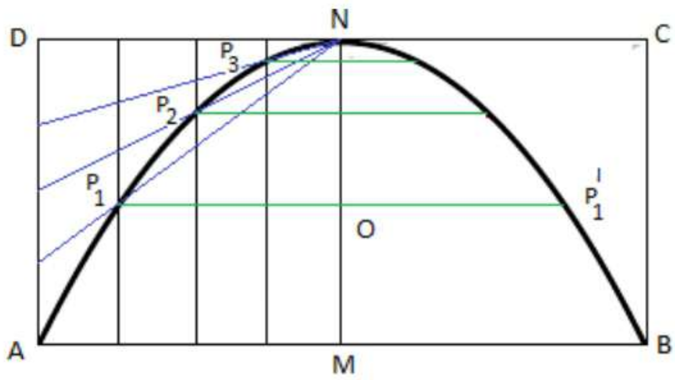


Figure 8

3. Tangent method

This method is also used when the base and the axis are given.

a. Draw a line AB as the base and mark the midpoint M on it.

b. Through M, draw the axis MN at right angles to AB.

c. Produce MN to O so that MN = NO.

d. Join O with A and B. Divide lines OA and OB into the same number of equal parts (say 7 or eight). More the number of parts better would be the parabola.

e. Mark the division points as shown in the figure.

f. Draw lines joining 11', 22', 33' and so on.

g. Draw a smooth curve starting from A and tangential to these lines. The curve so obtained is the required parabola.

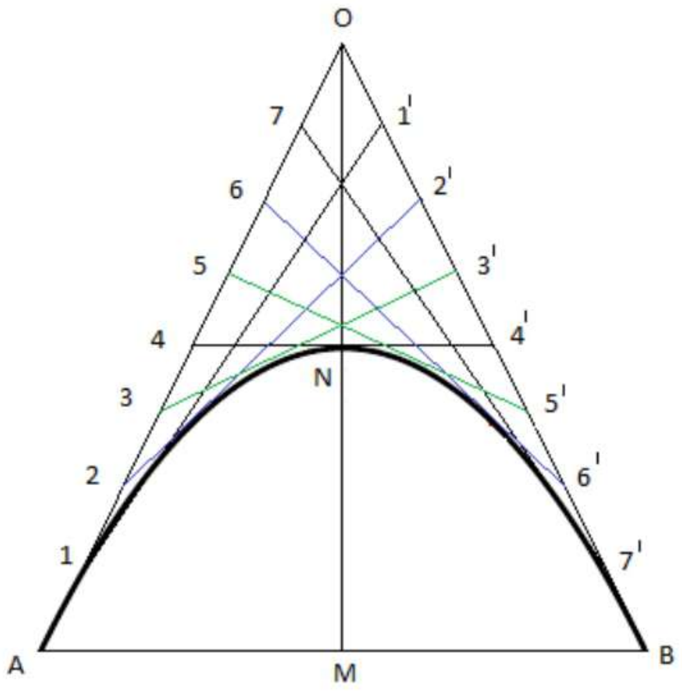


Figure 9

<u>Methods of construction of hyperbola</u>

1. Eccentricity method

This method is used when the distance of the focus from the directrix and the eccentricity are given. In the present case let us assume the eccentricity, e = 3/2

a. Draw a vertical line DD' as the directrix.

b. Draw CC' as the axis of the hyperbola from any suitable point C on the directrix such that CC' is perpendicular to the directrix.

c. Mark focus F on the axis CC'.

d. Divide CF into 5 equal parts and mark the vertex as V on the 2nd division point from C. Thus $eccentricity = \frac{VF}{VC} = 3/2$

e. Draw a perpendicular VE equal to VF. Now draw a line joining C and E and prolong it in the direction of CE. Thus , in triangle, CVE, $\frac{VE}{VC} = \frac{VF}{VC} = \frac{3}{2}$ (since VE = VF)

f. Mark a point 1 on the axis and draw a perpendicular through it so as to meet prolonged CE at $1'$.

g. With F as centre and radius equal to $1 - 1'$, draw arcs to intersect the perpendicular through 1 at P_1 and P_1'.

The points P_1 and P_1' lie on the required ellipse because the ratio P_1 from DD' is equal to C1, $P_1F = 1 - 1'$ and $\frac{1-1'}{C1} = \frac{VF}{VC} = \frac{3}{2}$. Similarly, mark points 2, 3 etc. on the axis and obtain points P_2 and P_2', P_3 and P_3' etc.

h. Draw the hyperbola through these points. The hyperbola so obtained is an open curve with a focus and a directrix.

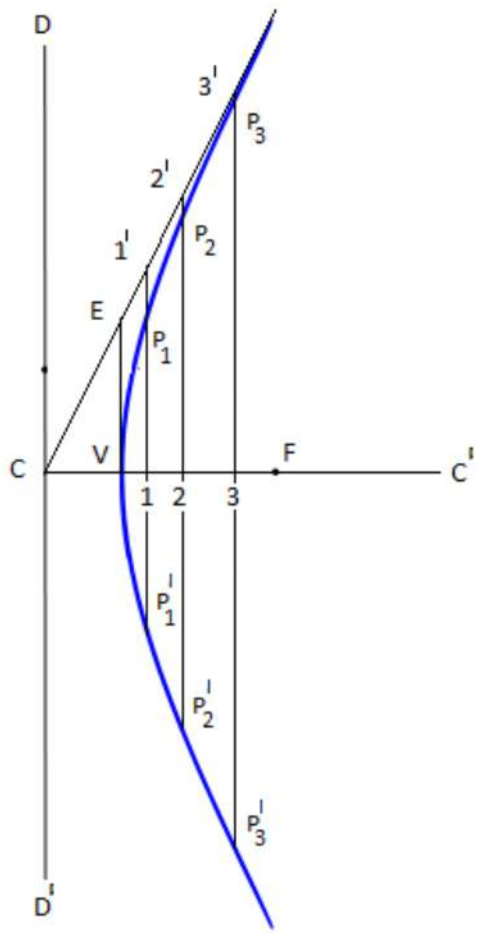


Figure 10

Examples on ellipse:

Problem 1

Construct an ellipse when the distance of the focus from the directrix is equal to 60 mm and eccentricity is 2/3.

Problem 2

A point moves in such a way its distance from a fixed line is always 1.5 times the distance from a fixed point 50 mm away from the fixed line. Draw the locus of the point and name the curve.

Problem 3

Draw an ellipse of major axis 150 mm if the ratio of the major axis to the minor axis is 3:2. Use concentric circle method.

Problem 4

A particle *P* moves such that the sum of its distances from two fixed points, 90 mm apart, remains constant. When *P* is at equal distance from the fixed points its distance from each one of them is 75 mm. Draw the path traced out by the particle. Hint: use arc of the circle method.

Solution:

a. Draw the major axis $AB = PF_1 + PF_2 = 2 \times 75 = 150\,mm$. P, is the position of particle at any instant.
b. Draw perpendicular bisector to AB at O.
c. Locate the foci at F_1 and F_2 45 mm each from the centre.
d. With F_1 as centre and PF_1 = 75 mm as radius, draw an arc to intersect the bisector through O on both sides of AB to get C and D.
e. Mark a number of points 1, 2, 3 . . . . etc. on the major axis AB between F_1 and F_2. Equidistant points from O make the construction convenient.
f. With F_1 and F_2 as centres and A1 as radius, draw arcs on both sides of AB.
g. With same centres and radius equal to B1, draw arcs intersecting the previous arcs at four points marked as P_1.
h. Similar points are obtained with radii A2 and B2; and A3 and B3 etc.
i. Draw a smooth curve through all these points. The curve, so obtained, is the required ellipse.

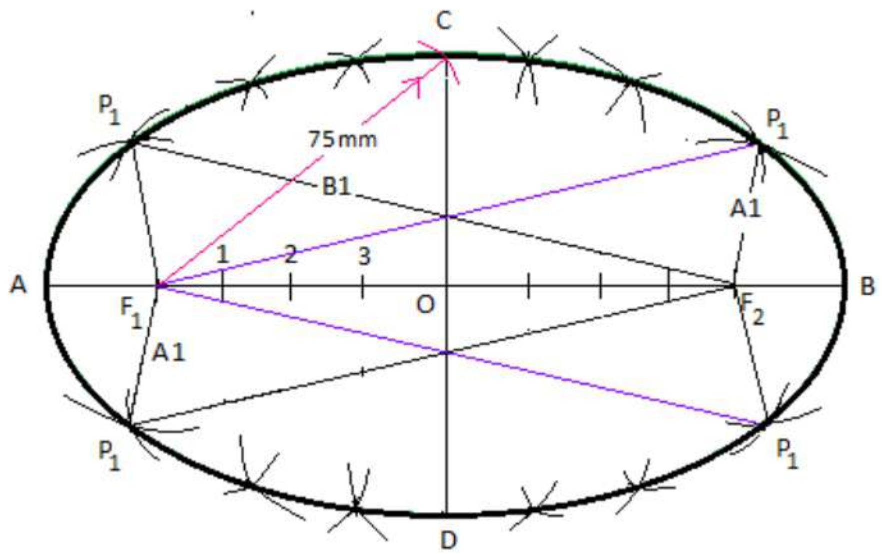


Figure 11

Examples on parabola:

Problem 1

Construct a parabola when the distance between the directrix and the focus is 50 mm.

Problem 2

Draw a parabola by tangent method, given the base and axis 60 mm and 30 mm respectively.

Problem 3

A cricket ball thrown up in the air reaches a maximum height of 10 m and falls on the ground at a distance of 30 m from the point of projection. Trace the path of the ball, assuming the path to be parabolic.

Examples on hyperbola:

Problem 1

Construct a hyperbola if the distance between the directrix and the focus is 25 mm and the eccentricity is 5;2.

Problem 2

Construct a hyperbola, given the distance between the focus and the directrix is 65 mm and the eccentricity is 3:2.

Ceiling of Statutory hall in US capitol

Roof of Skydome in Toronto

Attic in La Pedrera, Barcelona, Spain

McDonnel Planetarium, St. Louis, MO