LECTURE NOTES

ON

CONTROL SYSTEM ENGINEERING

6th Semester/ Electrical engineering

Department of Electrical Engineering

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COURSE CONTENTS

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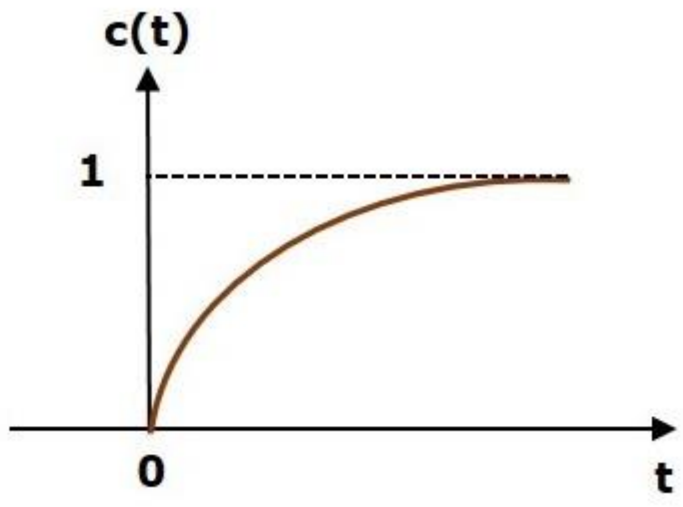
CHAPTER III

Analysis of stability by Root Locus Technique

What is Stability?

A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable. A **stable system** produces a bounded output for a given bounded input.

The following figure shows the response of a stable system.



This is the response of first order control system for unit step input. This response has the values between 0 and 1. So, it is bounded output. We know that the unit step signal has the value of one for all positive values of **t** including zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

Types of Systems based on Stability

We can classify the systems based on stability as follows.

- Absolutely stable system
- Conditionally stable system
- Marginally stable system

Absolutely Stable System

If the system is stable for all the range of system component values, then it is known as the **absolutely stable system**. The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of **'s' plane**. Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the 's' plane.

Conditionally Stable System

If the system is stable for a certain range of system component values, then it is known as **conditionally stable system**.

Marginally Stable System

If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as **marginally stable system**. The open loop control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis. Similarly, the closed loop control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginary axis.

Condition for stability

Let us consider a transfer function of a closed loop system:

$$\frac{C(s)}{R(s)} = \frac{a_0\, s^m + a_1\, s^{m-1} + a_2 s^{m-2} + \ldots + a_{m-1}\, s^1 + a_m s^0}{a_0\, s^n + a_1\, s^{n-1} + a_2\, s^{n-2} + \ldots + a_{n-1}\, s^1 + a_n\, s^0};$$

Here the characteristics Equation : $a_0\, s^n + a_1\, s^{n-1} + a_2\, s^{n-2} + \ldots + a_{n-1}\, s^1 + a_n\, s^0 = 0$

Necessary and sufficient conditions for stability:

1. All the coefficients of the ch. Equation should have same sign.

2. There should be no missing term.

Routh-Hurwitz Stability Criterion

This criterion is based on ordering the coefficients of the characteristics equation into an array called Routh's array. The Routh's array is formed as follows.

Follow this procedure for forming the Routh table.

- Fill the first two rows of the Routh array with the coefficients of the characteristic polynomial as mentioned in the table. Start with the coefficient of s_n and continue up to the coefficient of s_0.
- Fill the remaining rows of the Routh array with the elements as mentioned in the table . Continue this process till you get the first column element of **row** s_0 is a_n. Here, a_n is the coefficient of s_0 in the characteristic polynomial.

Note − If any row elements of the Routh table have some common factor, then you can divide the row elements with that factor for the simplification will be easy.

Consider the characteristic equation of the order 'n' is -

$a_0\, s^n + a_1\, s^{n-1} + a_2\, s^{n-2} + \ldots + a_{n-1}\, s^1 + a_n\, s^0 = 0$

s^n	a_0	a_2	a_4	a_6	…	…
s^{n-1}	a_1	a_3	a_5	a_7	…	…
s^{n-2}	$b1=(a1a2-a_3a0)/a1$	$b2=(a1a4-a5a0)/a1$	$b3=(a1a6-a7a0)/a1$	…	…	…
s^{n-3}	$c1=(b1a3-b2a1)/b1$	$c2=(b1a5-b3a1)/b1$	⁞⁞			

⁝	⁝	⁝	⁝			
S^1	⁝	⁝				
S^0	an					

Sufficient Condition for Routh-Hurwitz Stability

The sufficient condition is that all the elements of the first column of the Routh array should have the same sign. This means that all the elements of the first column of the Routh array should be either positive or negative.

Example

Let us find the stability of the control system having characteristic equation,

$S^4+3s^3+3s^2+2s+1=0$

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the characteristic polynomial, $S^4+3s^3+3s^2+2s+1$ are positive. So, the control system satisfies the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

S^4	1	3	1
S^3	3	2	
S^2	$\frac{(3\times3)-(2\times1)}{3} = 7/3$	$\frac{(3\times1)-(0\times1)}{3} = 1$	
S^1	$\frac{(7/3\times2)-(1\times3)}{7/3} = 5/7$		
S^0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability.

All the elements of the first column of the Routh array are positive. There is no sign change in the first column of the Routh array. So, the control system is stable.

Special Cases of Routh Array

The two special cases are −

- The first element of any row of the Routh array is zero.
- All the elements of any row of the Routh array are zero.

Let us now discuss how to overcome the difficulty in these two cases, one by one.

First Element of any row of the Routh array is zero

If any row of the Routh array contains only the first element as zero and at least one of the remaining elements have non-zero value, then replace the first element with a small positive integer, ϵ. And then continue the process of completing the Routh table. Now, find the number of sign changes in the first column of the Routh table by substituting ϵ tends to zero.

Example

Let us find the stability of the control system having characteristic equation,

$S^4+2s^3+s^2+2s+1=0$

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the characteristic polynomial, $S^4+2s^3+s^2+2s+1$are positive. So, the control system satisfied the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

S^4	1	1	1
S^3	2	2	
S^2	0	1	
S^1			
S^0			

Special case (i) − Only the first element of row s^2 is zero. So, replace it by ϵ and continue the process of completing the Routh table

.

s4	1	1	1
s3	1	1	
s2	ϵ	1	
s1	$[(\epsilon\times1)-(1\times1])/\epsilon = (\epsilon-1)/\epsilon$		
s0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability.

As ϵ tends to zero, the Routh table becomes like this.

s4	1	1	1
s3	1	1	
s2	0	1	

s1	-∞		
s0	1		

There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

All the Elements of any row of the Routh array are zero

In this case, follow these two steps −

- Write the auxilary equation, A(s) of the row, which is just above the row of zeros.
- Differentiate the auxiliary equation, A(s) with respect to s. Fill the row of zeros with these coefficients.

Example

Let us find the stability of the control system having characteristic equation,

$S^5+3s^4+s^3+3s^2+s+3=0$

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the given characteristic polynomial are positive. So, the control system satisfied the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s5	1	1	1
s4	$\cancel{3}$ 1	$\cancel{3}$ 1	$\cancel{3}$ 1
s3	0	0	
s2			
s1			
s0			

The row s4 elements have the common factor of 3. So, all these elements are divided by 3.
Special case (ii) − All the elements of row s3 are zero. So, write the auxiliary equation, A(s) of the row s4.
$A(s)=s^4+s^2+1$

Differentiate the above equation with respect to s.

$\frac{dA(s)}{ds} = 4s3+2s$

Place these coefficients in row s^3.

s5	1	1	1

s4	1	1	1
s3	4	2	
s2	0.5	1	
s1	−3		
s0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability.

There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

In the Routh-Hurwitz stability criterion, we can know whether the closed loop poles are in on left half of the 's' plane or on the right half of the 's' plane or on an imaginary axis. So, we can't find the nature of the control system. To overcome this limitation, there is a technique known as the root locus. We will discuss this technique in the next two chapters.

Root Locus

In the root locus diagram, we can observe the path of the closed loop poles. Hence, we can identify the nature of the control system. In this technique, we will use an open loop transfer function to know the stability of the closed loop control system.

The Root locus is the locus of the roots of the characteristic equation by varying system gain K from zero to infinity.

Angle Condition and Magnitude Condition

The points on the root locus branches satisfy the angle condition. So, the angle condition is used to know whether the point exist on root locus branch or not. We can find the value of K for the points on the root locus branches by using magnitude condition. So, we can use the magnitude condition for the points, and this satisfies the angle condition.

Characteristic equation of closed loop control system is

$1+G(s)H(s)=0$

$\Rightarrow G(s)H(s) = -1 + j0$

The **phase angle** of G(s)H(s) is
$\angle G(s)H(s)=\tan^{-1} 0/(-1)=(2n+1)\pi$

The **angle condition** is the point at which the angle of the open loop transfer function is an odd multiple of 180^0.

Magnitude of G(s)H(s) is -
$|G(s)H(s)| = \sqrt{(-1)^2 + 0^2} = 1$

The magnitude condition is that the point (which satisfied the angle condition) at which the magnitude of the open loop transfer function is one.

Rules for Construction of Root Locus

Follow these rules for constructing a root locus.

Rule 1 − Locate the open loop poles and zeros in the 's' plane.

Rule 2 − Find the number of root locus branches.

We know that the root locus branches start at the open loop poles and end either at open loop zeros or at ∞. So, the number of root locus branches **N** is equal to the number of finite open loop poles **P** or the number of finite open loop zeros **Z**, whichever is greater.

Mathematically, we can write the number of root locus branches **N** as

$N=P$ if $P \geq Z$
$N=Z$ if $P<Z$

Rule 3 − Identify and draw the **real axis root locus branches**.

A point or segment on the real axis lies on the root locus if the sum of open loop poles and open loop zeros to the right of this point or segment is odd.

Rule 4 − Find the centroid and the angle of asymptotes.

Asymptotes give the direction of these root locus branches.
Number of Asymptotes= P -Z
The intersection point of asymptotes on the real axis is known as **centroid**.

We can calculate the **centroid** $\boldsymbol{\sigma_A}$ by using this formula,

$$\boldsymbol{\sigma_A} = \frac{\sum \text{Real part of finite open loop poles} - \sum \text{Real part of finite open loop zeros}}{P-Z}$$

The formula for the angle of **asymptotes** is

$$\Phi_A = \frac{(2q+1)180}{P-Z}$$

Where,

$q=0,1,2,....,(P-Z-1)$

Rule 5 − Find the intersection points of root locus branches with an imaginary axis.

We can calculate the point at which the root locus branch intersects the imaginary axis and the value of **K** at that point by using the Routh array method

Rule 6 − Find Break-away and Break-in points.

- If there exists a real axis root locus branch between two open loop poles, then there will be a **break-away point** in between these two open loop poles.
- If there exists a real axis root locus branch between two open loop zeros, then there will be a **break-in point** in between these two open loop zeros.

Note − Break-away and break-in points exist only on the real axis root locus branches.

Follow these steps to find break-away and break-in points.

- Write K in terms of s from the characteristic equation $1+G(s)H(s)=0$

- Differentiate K with respect to s and make it equal to zero. Substitute these values of s in the above equation.
- The values of s for which the K value is positive are the **break points**.

Rule 7 − Find the angle of departure and the angle of arrival.

The Angle of departure and the angle of arrival can be calculated at complex conjugate open loop poles and complex conjugate open loop zeros respectively.

The formula for the **angle of departure** ϕ_d is
Φ_d = 180 –sum of the angles of vectors drawn to this pole to other poles + sum of the angles of vectors drawn to this pole to zeros

The formula for the **angle of arrival** φ_a is

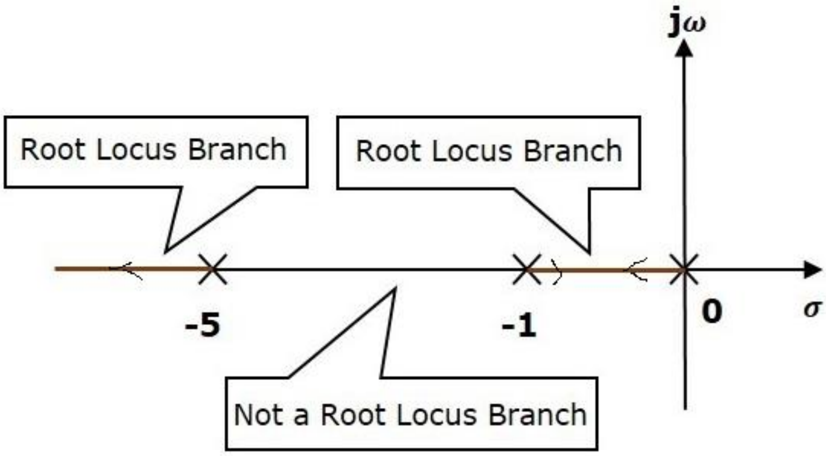
Φ_a = 180 –sum of the angles of vectors drawn to this zero to other zeros + sum of the angles of vectors drawn to this zero to poles

Example

Let us now draw the root locus of the control system having open loop transfer function
G(s)H(s)= $\frac{k}{s(s+1)(s+5)}$

Step 1 − The given open loop transfer function has three poles at s=0, s=−1 and s=−5. It doesn't have any zero. Therefore, the number of root locus branches is equal to the number of poles of the open loop transfer function.

N=P=3



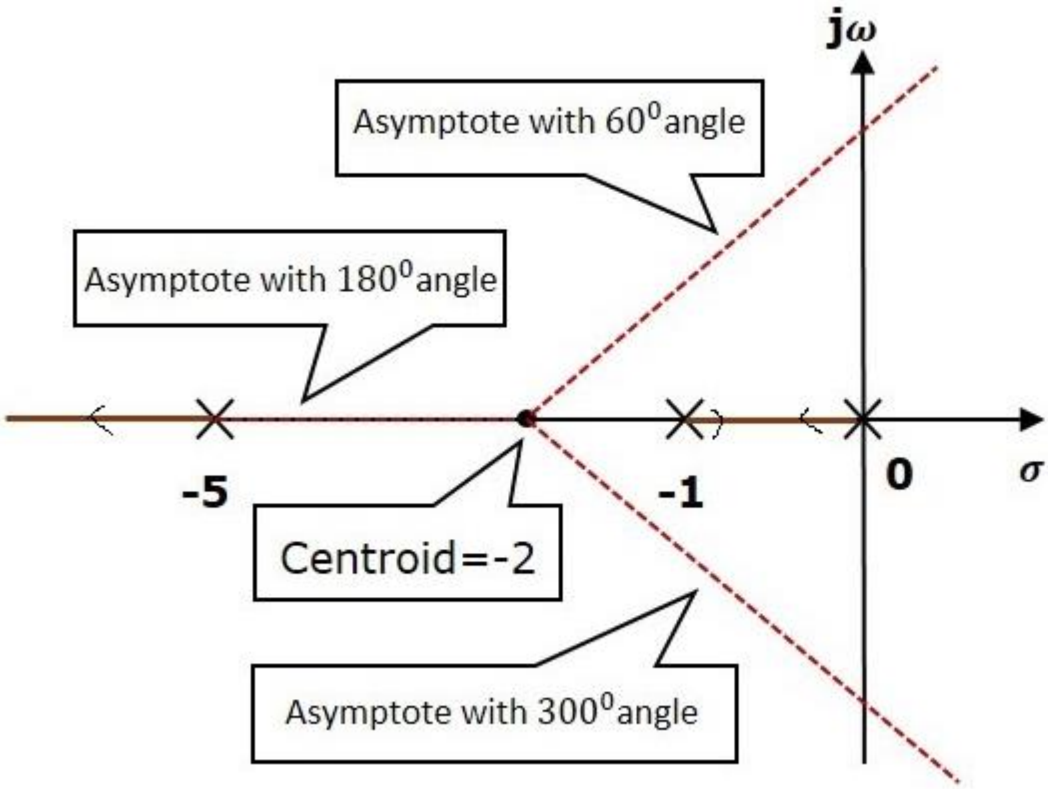
The three poles are located are shown in the above figure. The line segment between s=−1 and s=0 is one branch of root locus on real axis. And the other branch of the root locus on the real axis is the line segment to the left of s=−5 i.e in between -5 and ∞.

Step 2 − We will get the values of the centroid and the angle of asymptotes by using the given formulae.

Centroid $\boldsymbol{\sigma_A} = \frac{0-1-5}{3-0} = -2$

The angle of asymptotes $\Phi_A = \frac{(2q+1)180}{P-Z} = \frac{(2q+1)180}{3-0}$ for q= 0, 1, 2 angle of asymptotes are θ=60°,180° and 300°

The centroid and three asymptotes are shown in the following figure.



Step 3 − Since two asymptotes have the angles of 60^o and 300^o, two root locus branches intersect the imaginary axis. By using the Routh array method and special case(ii), the intersects of root locus branches to the imaginary axis can be found out as below

The characteristics equation of the given TF is 1+G(s)H(s)= 0

$$\text{Or } 1+\frac{k}{s(s+1)(s+5)} = 0$$

$$\text{Or } s^3+6s^2+5s+K = 0$$

Routh array

s3	1	5
s2	6	k
s1	$\frac{30-k}{6}$	0
s0	k	

For system stability the coefficient of Routh's array having positive and non zero value hence:

K > 0

$\frac{30-k}{6} > 0$ or k< 30

The range of K for which the system became stable is 0< k <30

At k = 30, the system auxiliary equation is

$6s^2 + 30= 0$

Or s = $\pm j\sqrt{5}$

Hence the root locus intersect the imaginary axis at $\pm j\sqrt{5}$

Step 4 − There will be one break-away point on the real axis root locus branch between the poles s=−1 and s=0. By following the procedure given for the calculation of break-away point,

The characteristics equation $s^3+6s^2+5s+K = 0$

Or $K= -(s^3+6s^2+5s)$

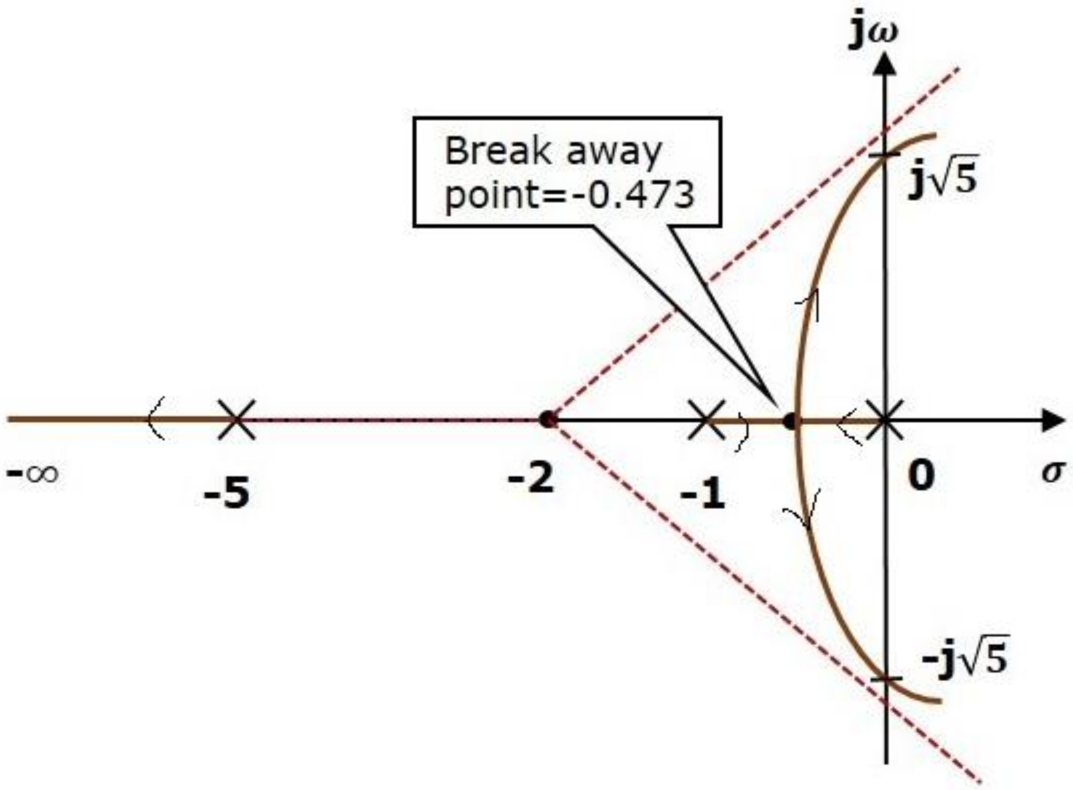
$\frac{dk}{ds} = 0$

Or $3s^2+12s+5=0$

The roots of s= -0.473, -3.52

Since breakaway point must lie between 0 and -1, it is clear that s=−0.473is actual breakaway point.

The root locus diagram for the given control system is shown in the following figure.



<u>Example</u> :- A feedback control system has open-loop transfer function

$$G(s)H(s) = \frac{K_a}{s(s+3)(S^2+2s+2)}$$

Draw root locus as K is varied from 0 to ∞ .

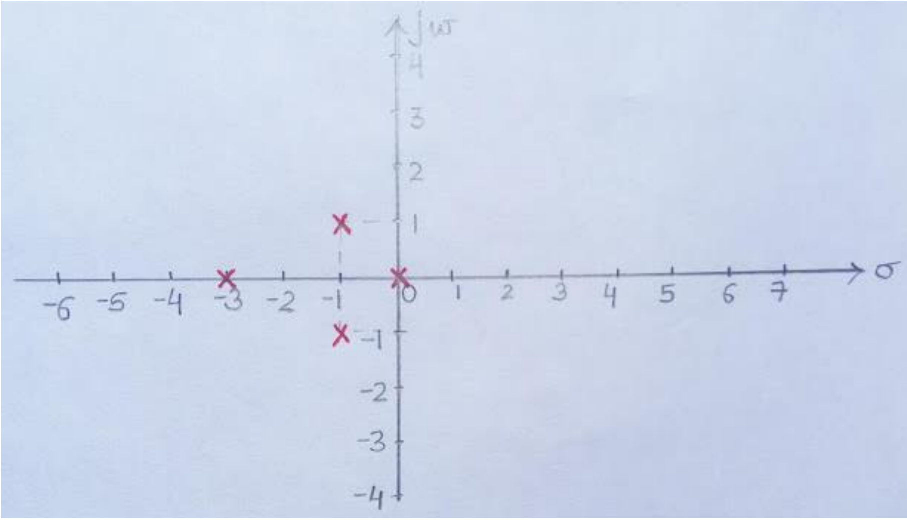
Solution:

<u>Step-1</u> :- Find OL poles and OL zeros from the OLTF

OL poles are S=0,-3, (-1+j1) and (-1-j1)

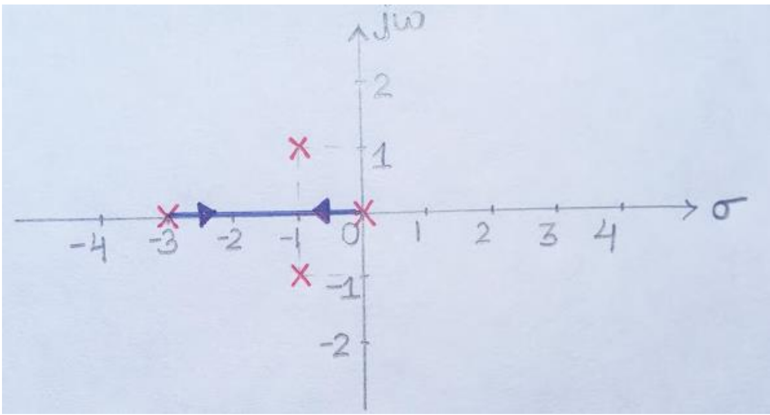
There are no finite OL zeros.

Mark OL pole with cross and OL zero with circle in S-plane as shown.



Step-2 Find the parts of the real axis at which root locus lies.

A point on real axis lies on root locus if the number of OL poles+OL zeros on the real axis to the right of the point is odd.Hence the Root locus exist between s=0 and s= -3 in the real axis.



Step-3 Number of root locus branches N = P= 4

Step-4 Find number of asymptotes:

Number of asymptotes = P - Z = 4 (where P,Z = nos of open loop pole and zero)

Step-5Calculation for cetroid

$$\boldsymbol{\sigma_A} = \frac{\sum \text{Real part of finite open loop poles} - \sum \text{Real part of finite open loop zeros}}{\text{P}-\text{Z}}$$

$$= \frac{(0-3-1-1)-(0)}{4} = -1.25$$

Step-6 Calculation for asymptotic angle:

$\Phi_A = \frac{(2q+1)180}{P-Z}$ For q=0; $\Phi_A = \frac{180\,(0+1)}{4} = 45^0$

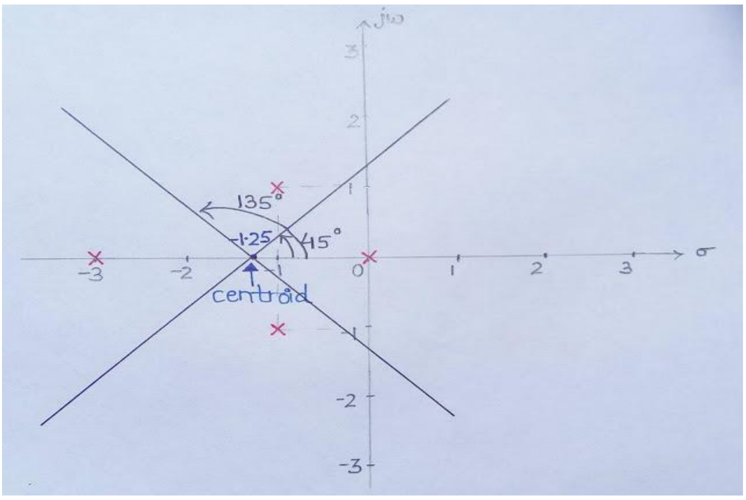
For q=1; $\Phi_A = \frac{180\,(2+1)}{4} = 135^0$

For q=2; $\Phi_A = 225^0$

$\Phi_A = 315^0$

For q=3;

So, from steps 2,3 and 4 , four asymptotes cut the real axis at -1.25 and make angles 45^0, 135^0, 225^0 and 315^0 , as shown below.



Step-7: Find the *breakaway points* (points at which two or more root locus branches meet)

Breakaway points are the solutions of $(dK_a/ds)=0$

The characteristic equation will be $S(S+3)(S^2+2S+2)+K_a=0$

From the characteristic equation,

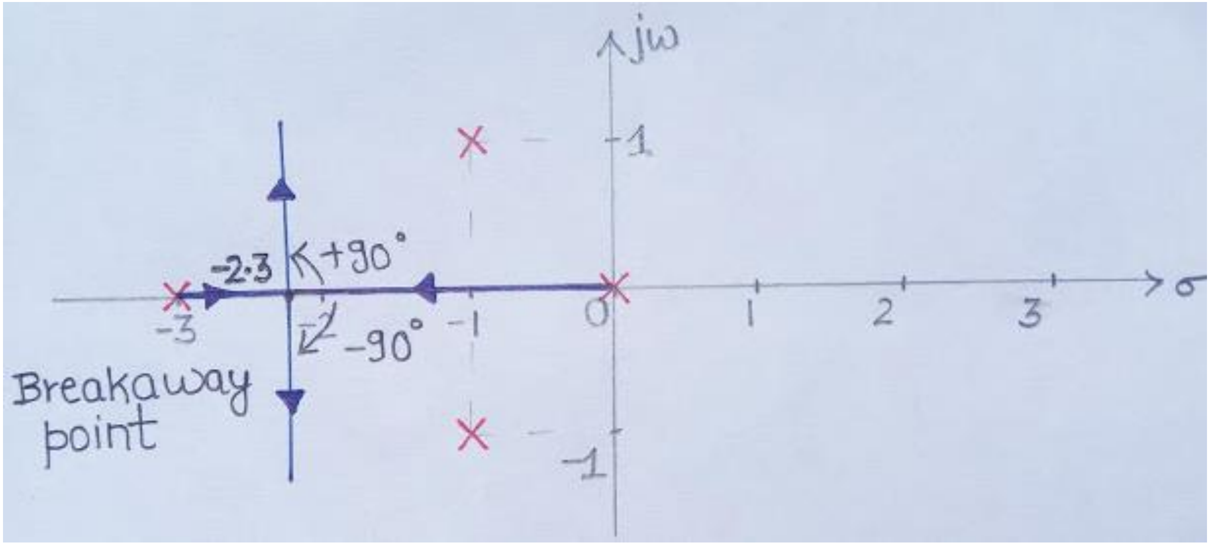
$K_a = -S(S+3)(S^2+2S+2) = -(S^4+5S^3+8S^2+6S)$

$$\therefore \frac{dK_a}{ds} = -4(S^3+3.75S^2+4S+1.5) = 0$$

We get, S = -2.3 , (-0.725±j0.365)

Not all values obtained as solutions of $(dK_a/ds)=0$ need to be necessarily the breakaway points. Out of the obtained s values only those values of S which satisfy angle condition are the actual breakaway points.

On checking angle condition we find that (-0.725±j0.365) do not satisfy it. Therefore, only S= -2.3 is the only breakaway point. So, the real axis from 0 to -3 contains root locus which breakdown at -2.3 as shown.



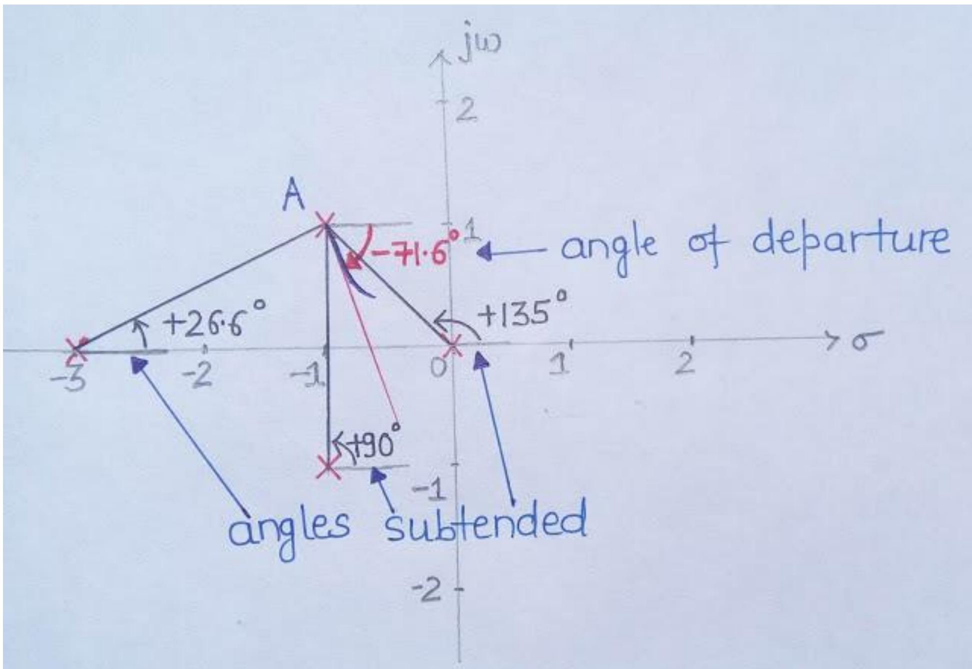
Step-8 :- Find *angles of departure* as there is a presence of pole in complex plane (angle which a root locus branch starting from an open loop pole, makes with a line parallel to the asymptotic line.

The formula for the **angle of departure** ϕ_d is

Φ_d = 180 –sum of the angles of vectors drawn to this pole to other poles + sum of the angles of vectors drawn to this pole to zeros

Or $\Phi_d = 180 - (90^0+135^0+26.6^0) = -71.6^0$

So, root locus branch starts from (-1+j1) at an angle -71.6^0 . Since root locus is always mirror image about real axis , therefore, root locus starts from (-1-j1) at $+71.6^0$.



Step-9 :- Find the points at which root locus branches intersect jw axis.

The characteristic equation will be $S(S+3)(S^2+2S+2)+K_a = 0$

Or $S^4+5S^3+8S^2+6S+ K_a=0$, Make rouths array;

S^4	1	8	K_a
S^3	5	6	
S^2	$\frac{(5\times8)-(6\times1)}{5} = 6.8$	K_a	
S^1	$\frac{(6.8\times6)-(Ka\times5)}{6.8}$		
S^0	K_a		

For the system to be stable all the coefficient of the first column of the Routh's array having positive and non zero value. Hence for system stability

$$K_a > 0$$

$$\frac{(6.8\times6)-(Ka\times5)}{6.8} > 0$$

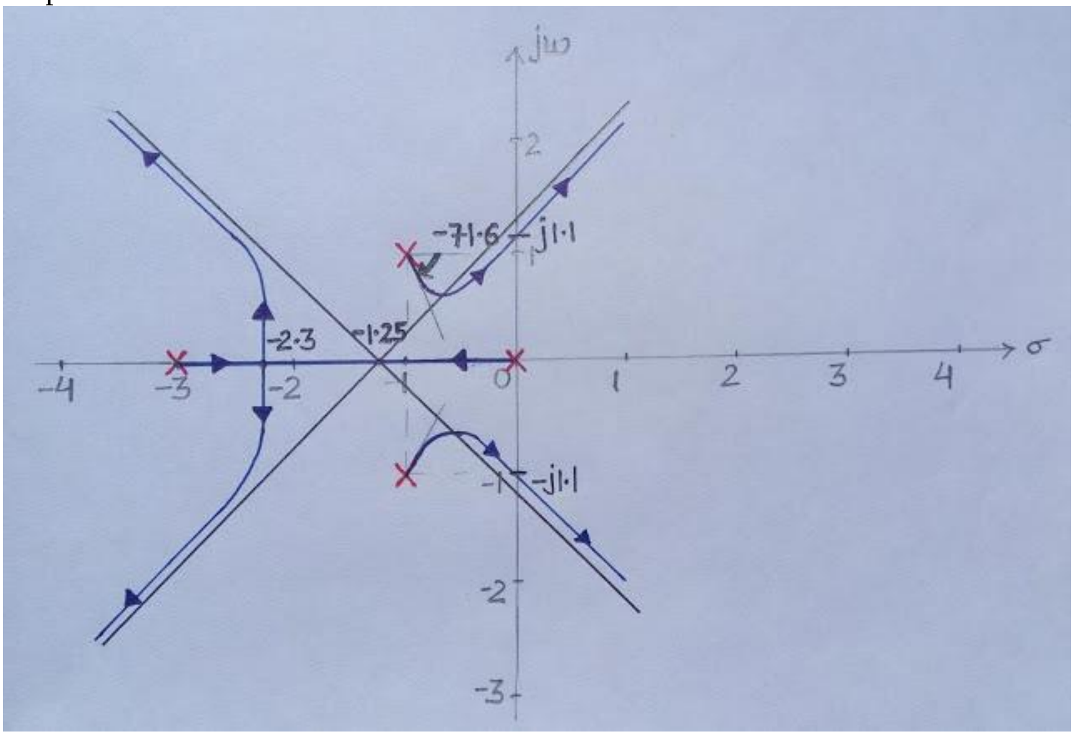
$$\text{Or } 0 < K_a < 8.16$$

For $K_a = 8.16$ the Auxiliary equation is $6.8s^2 + 8.16 = 0$

$$\text{Or } s^2 = -1.2$$

$$\text{Or } s = \pm j1.1$$

The points of intersection comes out to be $+j1.1$ and $-j1.1$
The complete root locus is shown below.



Effects of Adding Open Loop Poles and Zeros on Root Locus

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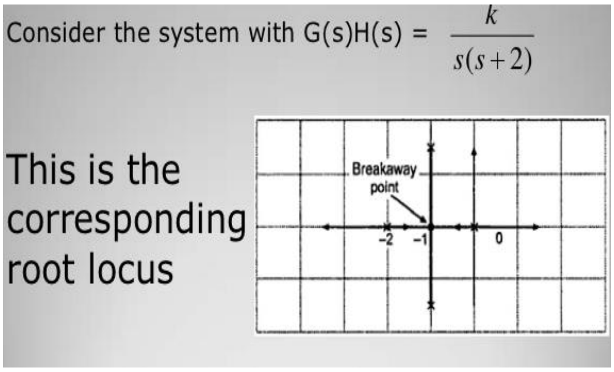

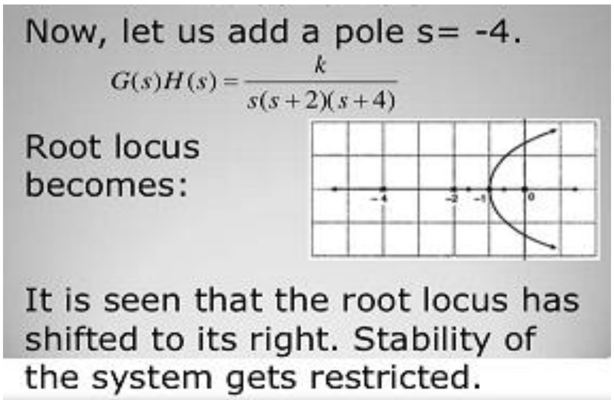

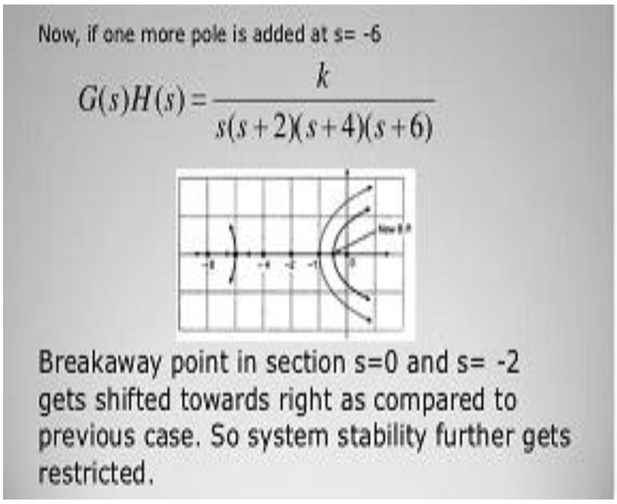

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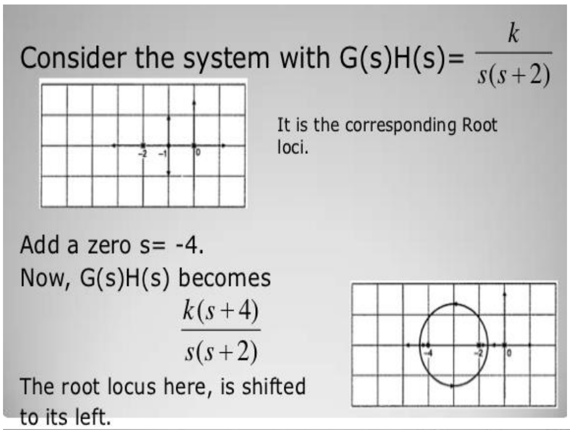

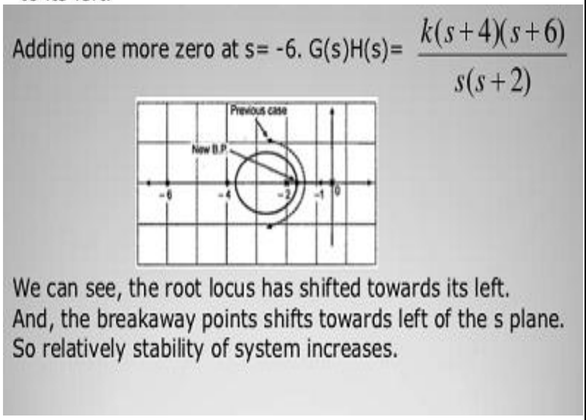

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➢The root locus changes its nature and shifts towards imaginary axis. ➢The system becomes oscillatory. ➢Gain Margin enhances relatively, thus stability decreases. ➢Range of k reduces. ➢Settling time increases.	➢The root locus changes its nature and shifts to left away from imaginary axis. ➢Relative stability of system increases. ➢System becomes less oscillatory. ➢Gain margin increases and so does the range of K. ➢Settling time decreases.