CHAPTER- V
Nyquist Plots

Introduction:

Nyquist plots are the continuation of polar plots for finding the stability of the closed loop control systems by varying ω from −∞ to ∞. That means, Nyquist plots are used to draw the complete frequency response of the open loop transfer function.

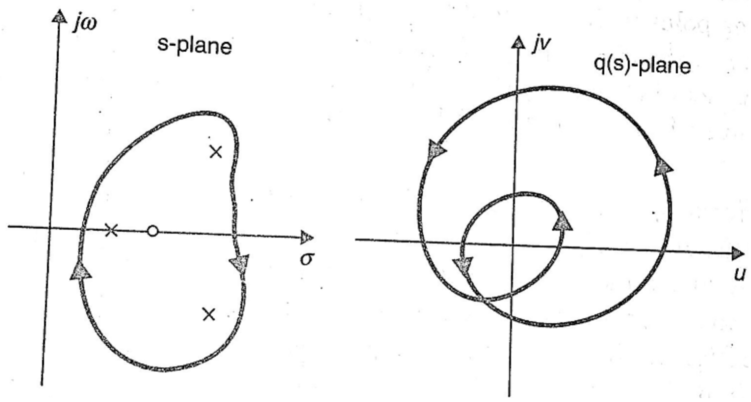
Principle of argument

The Nyquist stability criterion works on the **principle of argument**. It states that if there are P poles and Z zeros are enclosed by the 's' plane contour, then the corresponding G(s)H(s) plane must encircle the origin P−Z times. So, we can write the number of encirclements N as,

N=P−Z

- If the 's' plane contour contains only poles, then the direction of the encirclement in the q(s) plane will be opposite (counter clock wise) to the direction of 's' plane contour.
- If the 's' plane contour contains only zeros, then the direction of the encirclement in the q(s) plane will be in the same (clock wise) direction as that of 's' plane contour.

For example, in case of 1 zero and 3 poles enclosed by the s- plane contour, the net encirclement of the origin by the q(s) plane contour is (3-1) two counter-clockwise revolution as shown in figure below. This relationship between the enclosure of poles and zeros of G(s)H(s) b the s-plane contour and the encirclement of the origin by G(s)H(s) contour is commonly known as principle of argument.



Nyquist stability criterion

The characteristics equation of a system is

q(s) = 1+G(s)H(s)

The standard pole zero form of the OLTF G(s)H(s) is

$$G(s)H(s) = K\,\frac{(s+z_1)(s+z_2)-------(s+z_m)}{(s+P_1)(s+P_2)--------(s+P_n)} \qquad (1)$$

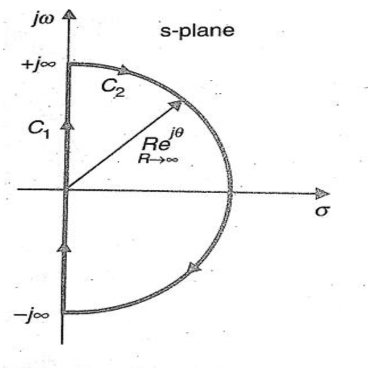
$$\therefore q(s) = 1+ K\frac{(s+z_1)(s+z_2)-------(s+z_m)}{(s+P_1)(s+P_2)--------(s+P_n)}$$

$$= \frac{(s+P_1)(s+P_2)--------(s+P_n)+K(s+z_1)(s+z_2)-------(s+z_m)}{(s+P_1)(s+P_2)--------(s+P_n)}$$

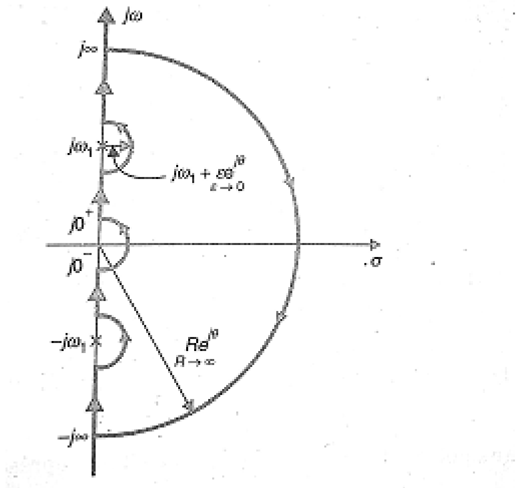
$$= \frac{(s+z_1')(s+z_2')---------(s+z_n')}{(s+P_1)(s+P_2)--------(s+P_n)} \qquad (2)$$

From the above equation it is seen that the zeros of q(s) are the root of the characteristics equation and the poles q(s) are same as the poles of open loop system. For the system to be stable, the roots of the characteristics equation and hence the zeros of q(s) must lie in the left half s-plane. It is important to note that even if some of the open-loop poles lie in the right half s-plane all the zeros of q(s) i.e, the closed-loop poles may lie in the left half s-plane. It means that an open-loop unstable system may lead to a closed-loop stable system.

In order to investigate the presence of any zero of q(s) in the right half of s-plane, a contour to be chosen which completely encloses the right half of s-plane called as **Nyquist contour**. It is directed clockwise and consist of an infinite line segment C_1 and an arc C_2 of infinite redius.



The Nyquist contour.



Indented Nyquist contour for jω-axis open-loop poles.

As the Nyquist contour encloses all the right half s-plane poles and zeros of q(s), let there are 'z' zeros and 'P' poles in the right half of s-plane. As s moves along the nyquist contour in the s-plane, a closed contour Γ_q is traversed in q(s) plane which encloses the origin N (=P-Z) times in anticlockwise direction.

For the system to be stable, there should be no zeros of q(s) in the right half of s-plane i.e,

$$Z = 0$$

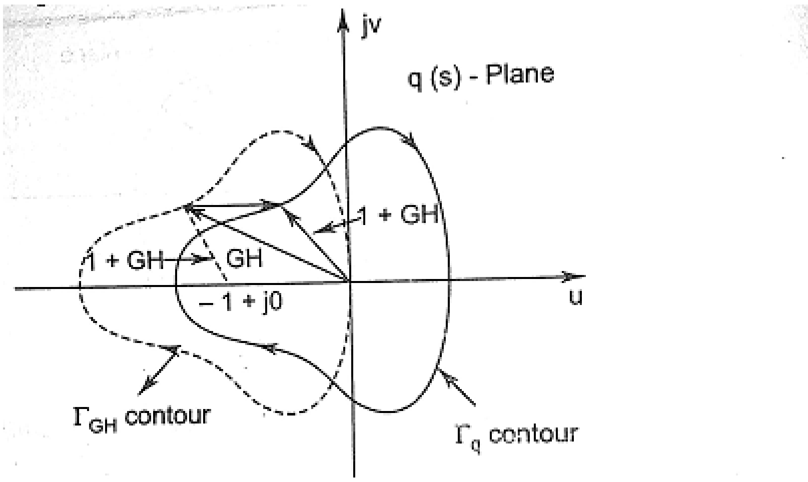
$$\text{So } N= P$$

The above equation implies that for a close loop system to be stable, the number of counter-clockwise encirclement of the origin of the q(s) plane by the contour Γ_q should be equal the number of the right half s-plane poles of q(s) which are also the poles of open-loop transfer function G(s)H(s).

The open-loop transfer function can be written as

$$G(s)H(s)=q(s)- 1= [1+ G(s)H(s)]-1 \qquad (3)$$

Therefore the contour Γ_{GH} of G(s)H(s) corresponding to the nyquist contour in the s-plane is the same as contour Γ_q of q(s) (=1+ G(s)H(s)) drawn from the point (-1+j0). Thus the encirclement of the origin by the contour Γ_q of q(s) is equivalent to the encirclement of the point (-1+j0) by the contour Γ_{GH} of G(s)H(s) as shown below.



Contour Γ_{GH} of G(s) H (s) corresponding to Nyquist contour.

Statement of nyquist stability criterion:

1. If the contour Γ_{GH} corresponding to the Nyquist contour in the s-plane encircles the point (-1+j0) in the counter-clockwise direction as many times as the number of right half s-plane pole of G(s)H(s), the close loop system is stable.
2. The closed loop system is stable if the contour Γ_{GH} does not encircles the point (-1+j0).

Mapping of Nyquist contour into the contour Γ_{GH} of G(s)H(s):

1. For imaginary axis: Put s= jω in G(s)H(s) where s varies from –j∞ to +j∞ .

2. For infinite semi circle: put s= $Re^{j\theta}$ where R→∞ and θ varies from + 90° to -90°.

3. For presence of pole at origin: put s= $\in e^{j\theta}$ where ∈→∞ and θ varies from - 90° to +90°.

4. For presence of pole at imaginary axis: put s= $j\omega_1 + \in e^{j\theta}$ where ∈→∞ and θ varies from - 90° to +90°.

Hence the complete contour Γ_{GH} is the polar plot of G(jω)H(jω) with varies from ω -∞ to +∞.

Nyquist stability criterion applied to inverse Polar plot:

It is more convenient to work with inverse function 1/ G(jω)H(jω) rather than the direct function G(jω)H(jω). Here we will see that the Nyquist stability criterion for direct polar plot can be extended for use to inverse polar plot after minor modification.

Let us consider a open-loop transfer function:

$$G(s)H(s) = K\frac{(s+z_1)(s+z_2)-------(s+z_m)}{(s+P_1)(s+P_2)--------(s+P_n)} \quad (4)$$

For the system to be stable none of the roots of the characteristics equation should lie in the right half s-plane or on the jω-axis. The characteristics equation is

$$q(s)= 1+ G(s)H(s) = \frac{(s+z_1')(s+z_2')---------(s+z_n')}{(s+P_1)(s+P_2)--------(s+P_n)} \qquad (5)$$

Dividing equation 5 by 4, we get

$$q'(s)= \frac{1}{G(s)H(s)} +1= \frac{(s+z_1')(s+z_2')---------(s+z_n')}{(s+z_1)(s+z_2)-------(s+z_m)} \qquad (6)$$

From equation 5 and 6 it is seen that the zeros of q'(s) is same as the q(s), which are the roots of the characteristics equation. It is further noticed that the poles of q(s) are same as the poles of G(s)H(s), while the poles of q'(s) are same as the poles of $\frac{1}{G(s)H(s)}$ or the zeros of G(s)H(s).

It can be concluded that if $\frac{1}{G(s)H(s)}$ has P right half s-plane poles and the characteristics equation has Z right half s-plane zeros, the locus of $\frac{1}{G(s)H(s)}$ encircle the point (-1+j0) N times in counter-clockwise direction where N= P-Z.

Since for system stability no zeros of the characteristics equation locate on right half s-plane i.e , Z=0, the Nyquist stability criterion for inverse polar plots can be stated below:

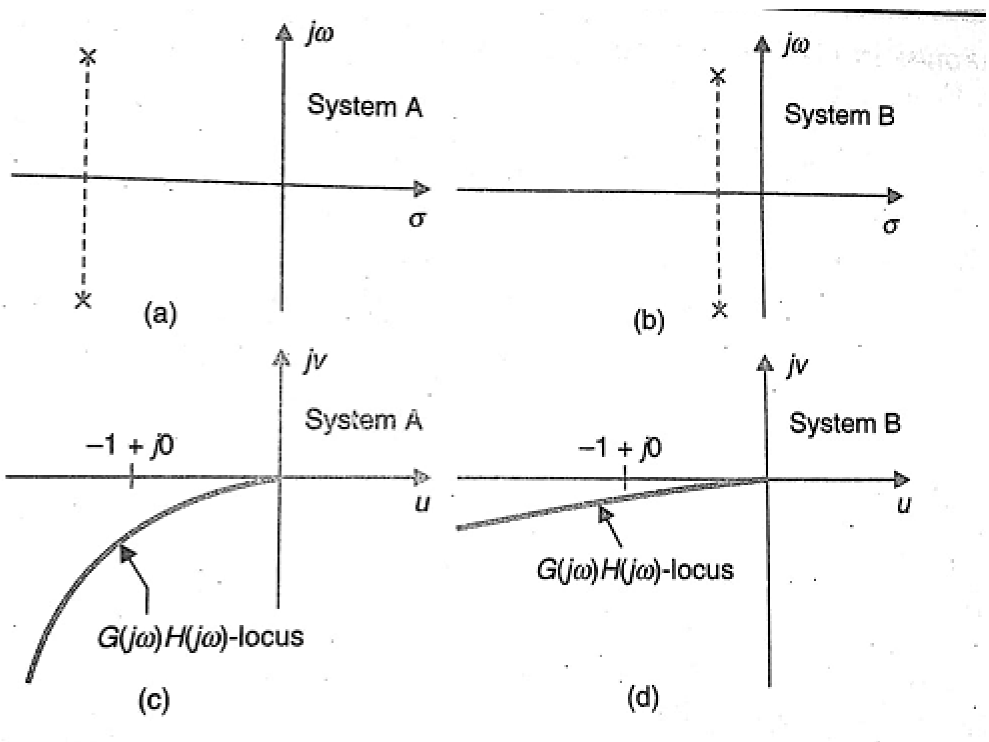
"It the Nyquist plot of $\frac{1}{G(s)H(s)}$ corresponding to the Nyquist contour in the s-plane, encircles counter-clockwise the point (-1+j0) as many times as are the number of right half s-plane pole of $\frac{1}{G(s)H(s)}$, the closed-loop system is stable. "

In special case where $\frac{1}{G(s)H(s)}$ has no pole in the right half s-plane, the close loop system is stable provided the net encirclement of (-1+j0) point by the Nyquist plot of $\frac{1}{G(s)H(s)}$ is zero.

Assessment of relative stability using Nyquist criterion:

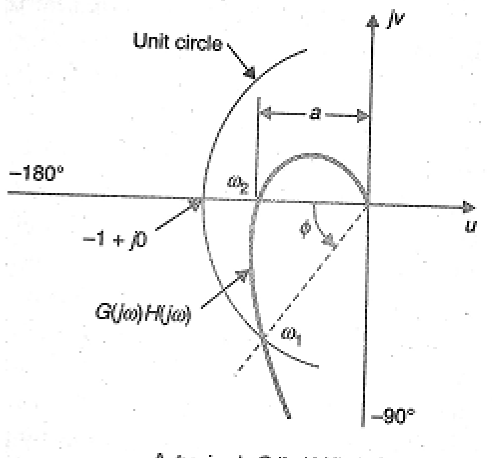
The measure of relative stability of a closed-loop systems which are open-loop stable can be analysed through the study of Nyquist plots. The stability of such system can be determined by polar plot of G(s)H(s). It can be imagined that as the polar plot gets closer to (-1+j0) point, the system tends towards instability.

Consider two different systems whose closed loop poles are shown on the s-plane in figure a and b respectively. It is seen that system A is more stable than system B because its closed-loop poles are located comparatively away to the left from jω-axis. The open-loop frequency response (polar) plots for system A and B are shown in figure 'c' and 'd', respectively. The comparison of the closed-loop pole location of these two system with their corresponding polar plot shows that as a polar plot moves closer to (-1+j0) point, the system closed-loop poles move closer to the jω-axis and hence the system becomes relatively less stable and vice versa.



Correlation between the closed-loop s-plane root locations and open-loop frequency response curves.

The figure as given below shows a $G(j\omega)H(j\omega)$ locus which crosses the negative real axis at a frequency $\omega=\omega_2$ with an intercept of a. Let a unit circle centred at origin (passes through point -1+j0) intersect the $G(j\omega)H(j\omega)$ locus at a frequency $\omega=\omega_1$ and let the phasor $G(j\omega_1)H(j\omega_1)$ makes an angle of ϕ with the negative real axis measured positively in counter-clockwise direction. It is observed that as $G(j\omega)H(j\omega)$ locus approaches (-1+j0) point, the relative stability reduces.



A typical $G(j\omega)H(j\omega)$-locus.

Constant Magnitude Loci or Constant M Circle

The closed loop transfer function of a unity feedback system is given by

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

or
$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)}$$

Let
$$G(j\omega) = x + jy$$

$\therefore$
$$T(j\omega) = \frac{(x+jy)}{1+(x+jy)} = \frac{x+jy}{1+x+jy} \quad (1)$$

Let magnitude of T $(j\omega)$ is M, we can write

$$M = \frac{|x+jy|}{|1+x+jy|} = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}} \quad (2)$$

Squaring both sides, we get

$$M^2 = \frac{x^2+y^2}{(1+x)^2+y^2} \quad (3)$$

or
$$M^2\left[(1+x)^2+y^2\right] = x^2+y^2$$

or
$$M^2\left[1+x^2+2x+y^2\right] = x^2+y^2$$

or
$$M^2\left[x^2+y^2+2x+1\right] - x^2 - y^2 = 0$$

or
$$x^2(M^2-1) + 2xM^2 + y^2(M^2-1) + M^2 = 0$$

or
$$x^2 + \frac{2x}{(M^2-1)}.M^2 + y^2 + \frac{M^2}{(M^2-1)} = 0$$

or
$$x^2 - \frac{2x}{(1-M^2)}.M^2 + y^2 - \frac{M^2}{(1-M^2)} = 0$$

or $$x^2 - \frac{2xM^2}{(1-M^2)} + y^2 = \frac{M^2}{(1-M^2)}$$

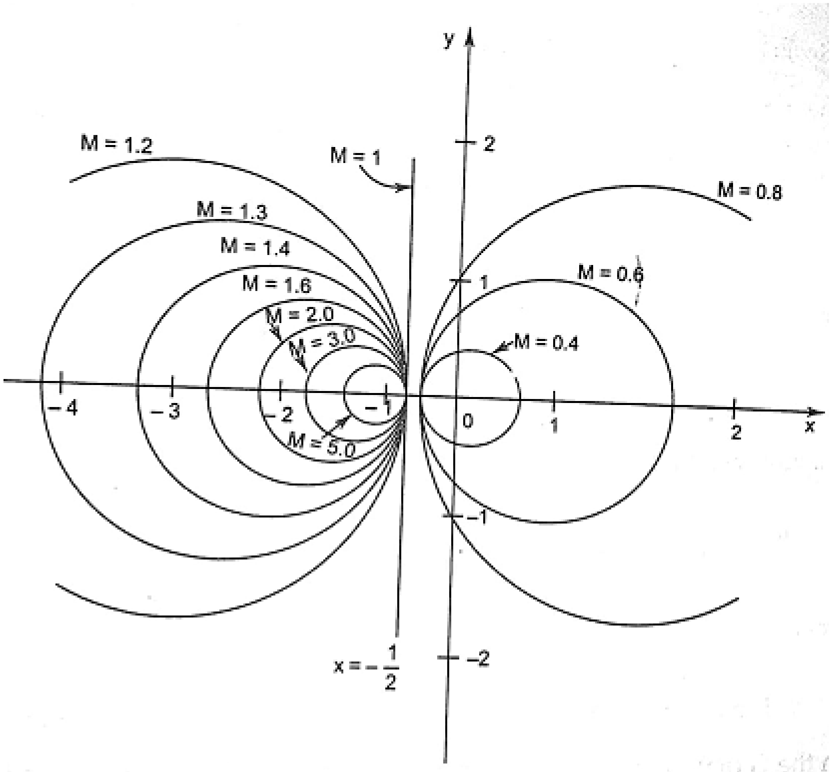
or $$x^2 - \frac{2xM^2}{(1-M^2)} + \left[\frac{M^2}{(1-M^2)}\right]^2 + y^2 = \frac{M^2}{(1-M^2)} + \left[\frac{M^2}{(1-M^2)}\right]^2$$

or $$\left[x - \frac{M^2}{(1-M^2)}\right]^2 + y^2 = \frac{M^2}{(1-M^2)^2} \qquad \textbf{(4)}$$

Equation 2 represents the equation of a circle with centre at $\left[\frac{M^2}{(1-M^2)}, 0\right]$ having radius of $\frac{M}{(1-M^2)}$

If M=1, then Equation 3 becomes$(1+x)^2 + y^2 = x^2 + y^2$ or x= $-\frac{1}{2}$ (5)

It is a equation for straight line parallel to the y-axis and passing through $\left(-\frac{1}{2}, 0\right)$ in the G(jω) plane. For each value of M (except M=1) we get a circle. These circles are known as Constant Magnitude Loci or Constant M Circle.



Constant M-circles.

Constant Phase Loci or Constant N Circle

From equation 1

$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{(x+jy)}{(1+x)+jy}$$

Phase angle of T (jω) is given by

$$\angle T(j\omega) = \angle\left[\frac{C(j\omega)}{R(j\omega)}\right] = \angle\left[\frac{(x+jy)}{(1+x)+jy}\right]$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{1+x}\right) \tag{6}$$

$$\Rightarrow \qquad \tan\theta = \frac{\frac{y}{x} - \frac{y}{1+x}}{1 + \frac{y^2}{x(1+x)}}$$

$$= \frac{y}{x^2 + x + y^2}$$

Let

$$\tan\theta = N \tag{7}$$

$$\therefore \qquad N = \frac{y}{x^2 + x + y^2} \tag{8}$$

or

$$N(x^2 + x + y^2) = y$$

or

$$x^2 + x + y^2 - \frac{y}{N} = 0$$

Add $\left(\frac{1}{4} + \frac{1}{4N^2}\right)$ to both sides, we get **(9)**

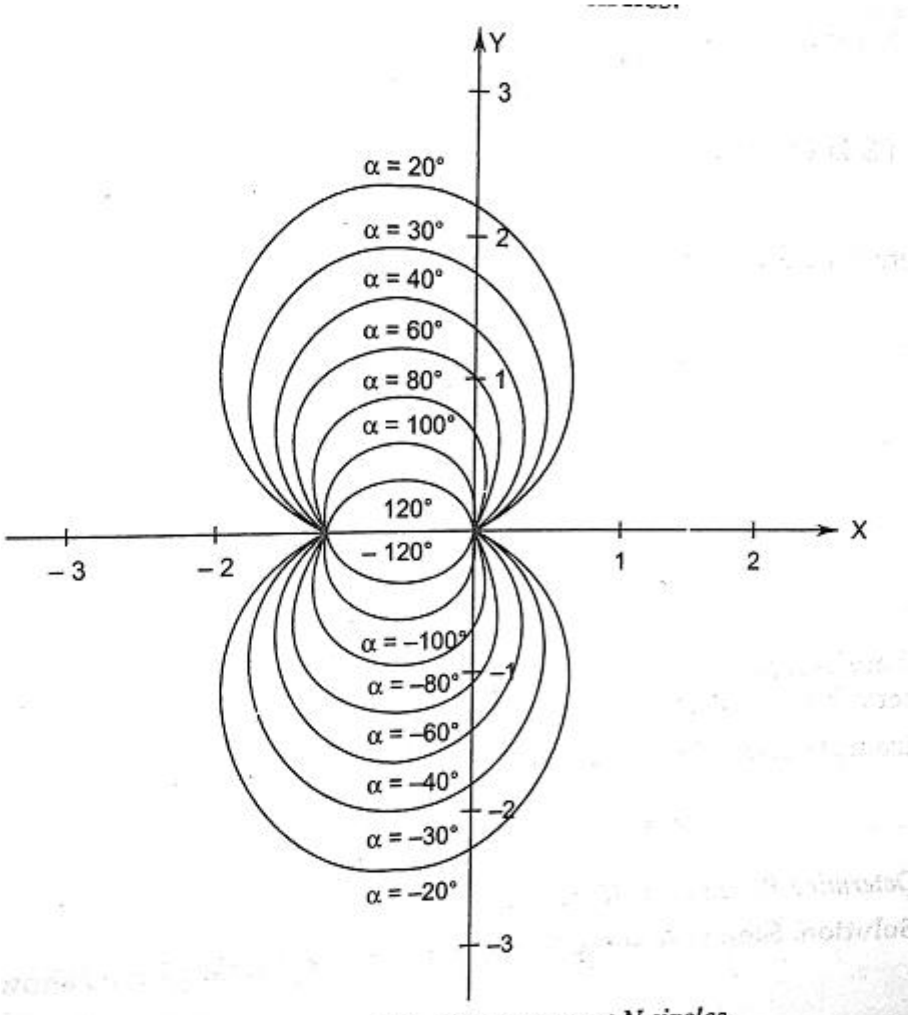
$$x^2 + x + y^2 - \frac{y}{N} + \frac{1}{4} + \frac{1}{(4N^2)} = \frac{1}{4} + \frac{1}{4N^2}$$

or

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{4N^2} \tag{10}$$

Equation 10 represents the equation of circle with it centre at $\left(-\frac{1}{2}, \frac{1}{2N}\right)$ with radius $\sqrt{\left(\frac{1}{4} + \frac{1}{4N^2}\right)}$

For different values of N i.e, phase angle θ, equation 10 represents the family of the circles. For a particular circle, the value of N i.e, phase angle θ remain constant on it. Therefore these circle are known as constant phase loci or N circles.



A family of constant N circles.

Nichols Plot

Constant magnitude loci that are M-circles and constant phase angle loci that are N-circles are the fundamental components in designing the Nichols chart. The constant M and constant N circles in G (jω) plane can be used for the analysis and design of control systems. However the constant M and constant N circles in gain phase plane are prepared for system design and analysis as these plots supply information with fewer manipulations. Gain phase plane is the graph having gain in decibel along the ordinate (vertical axis) and phase angle along the abscissa (horizontal axis). The M and N circles of G (jω) in the gain phase plane are transformed into M and N contours in rectangular co-ordinates. A point on the constant M loci in G (jω) plane is transferred to gain phase plane by drawing the vector directed from the origin of G (jω) plane to a particular point on M circle and then measuring the length in db and angle in degree. The critical point in G (jω), plane corresponds to the point of zero decibel and -180^o in the gain phase plane. Plot of M and N circles in gain phase plane is known as Nichols chart /plot.

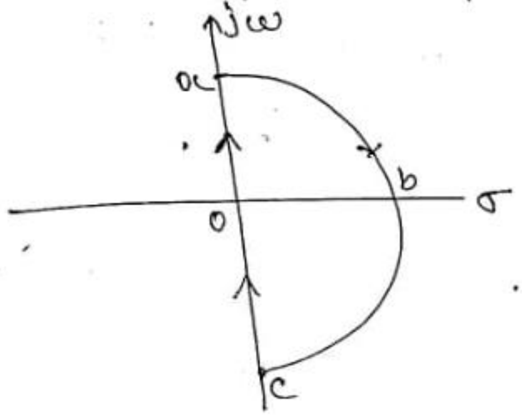
The **Nichols plot** is named after the American engineer N.B Nichols who formulated this plot. Compensators can be designed using Nichols plot. Nichols plot technique is however also used in designing of dc motor. This is used in signal processing and control design. Nyquist plot in complex plane shows how phase of transfer function and frequency variation of magnitude are related. We can find out the gain and phase for a given frequency. Angle of positive real axis determines the phase and distance from origin of complex plane determines the gain.

Example: 1 - Sketch Nyquist plot for a given OLTF.

$$G(s)H(s) = \frac{K}{(T_1 s+1)(T_2 s+1)}$$

Solution

Step 1: First draw nyquist path for the given OLTF.



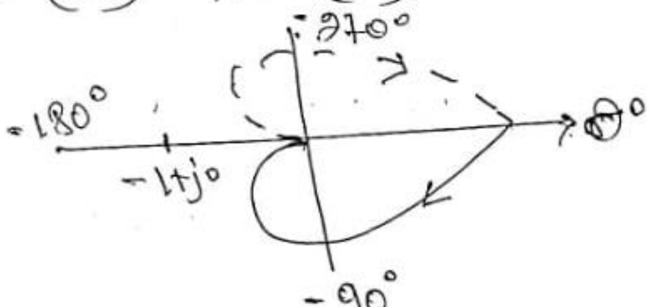
Step 2: For path 0a put $s=j\omega$ where ω varies from 0 to ∞ & plot polar plot.

$$G(j\omega)H(j\omega) = K / (T_1 j\omega+1)(T_2 j\omega+1)$$

$$M = |G(j\omega)H(j\omega)| = K / [\{(T_1\omega)^2+1\}\{(T_2\omega)^2+1\}]$$

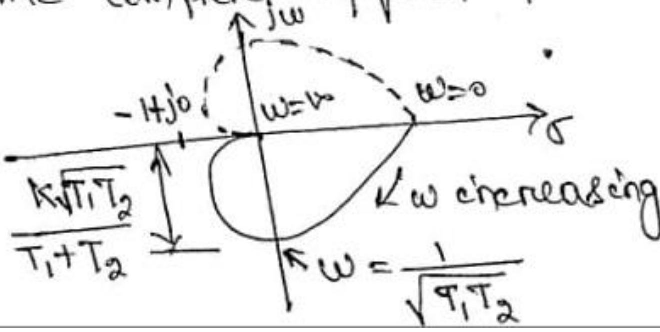
$$\phi = \angle G(j\omega)H(j\omega) = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

ω	M	φ
0	K	0°
∞	0	-180°



Step 3: Path co is the mirror image of path oa shown by a doted line in the figure as given below,

Step 4: The complete nyquist plot is:

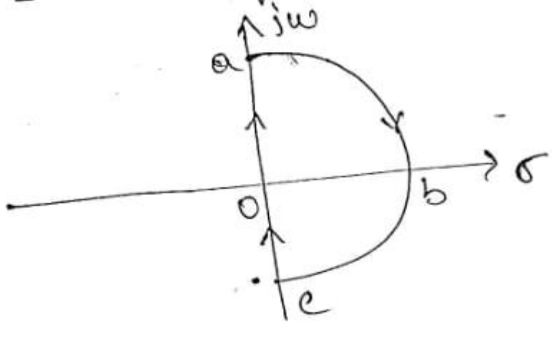


Step 5: It is seen that the plot of $G(j\omega)$ does not encircle the point $(-1+j0)$ i.e $N=0$, therefore the system is stable.

Example 2: Sketch Nyquist plot for the given OLTF.

$$G(s)H(s) = \frac{s+2}{(s+1)(s-1)}$$

Sol^n Step 1: Sketch Nyquist path for the OLTF.



Step 2: For path oa put $s=j\omega$ & sketch polar plot.

$$G(j\omega)H(j\omega) = \frac{2+j\omega}{(1+j\omega)(-1+j\omega)}$$

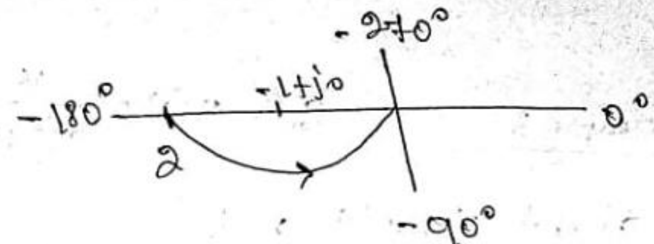
$$M = |G(j\omega)H(j\omega)| = \frac{\sqrt{4+\omega^2}}{\sqrt{1+\omega^2}\sqrt{1+\omega^2}}$$

$$\phi = \angle G(j\omega)H(j\omega) = \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{1} - \tan^{-1}(-\omega)$$

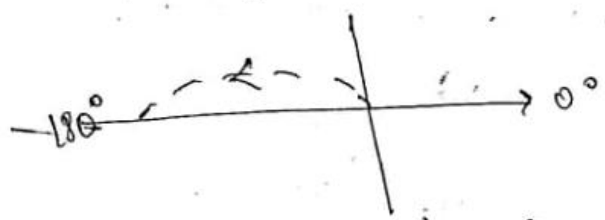
$$= \tan^{-1}\frac{\omega}{2} - \tan^{-1}\omega - (180 - \tan^{-1}\omega)$$

ω	M	ϕ
0	2	-180°
∞	0	-90°

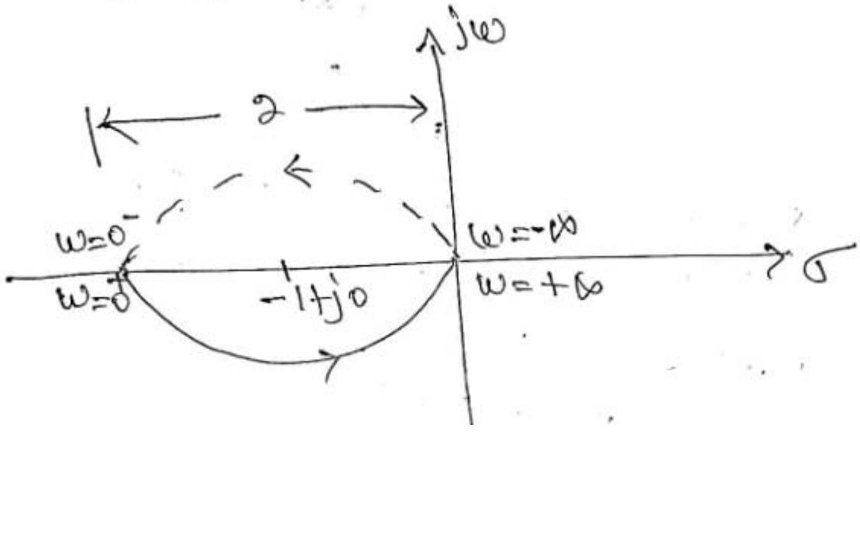
Polar plot is



Step 3: Path CO is the mirror image of the path oa, as shown below:



Step 4: The complete contour for $F_1(j\omega)$ is as given below.



Step 5: Check for stability

From the OLTF no. of pole present in the right half of s-plane $P = 1$.

From Nyquist Plot no of counter clockwise encirclement $N = 1$.

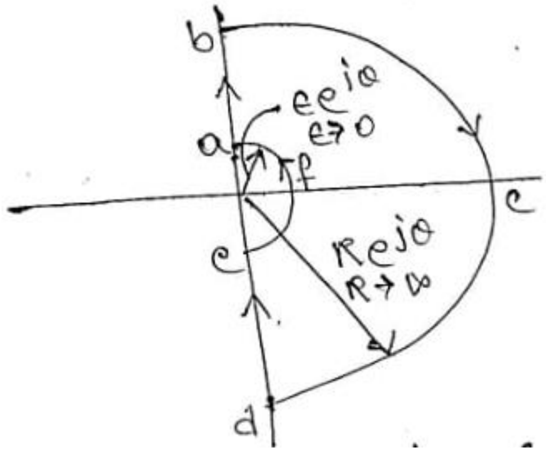
$\therefore N = P - Z \Rightarrow 1 = 1 - Z \Rightarrow Z = 0$

which states that there are no zeros of $1+G(s)H(s)$ in the right half s-plane and hence the closed-loop system is stable.

Example 3 $\quad G(s)H(s) = \dfrac{K}{s(Ts+1)}$

Sketch Nyquist Plot?

Soln Step1: As the given OLTF has pole at origin the Nyquist contour is as given below:



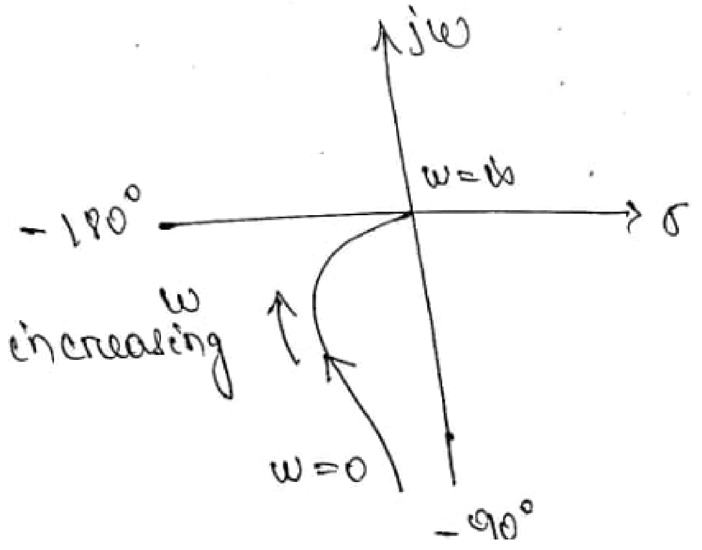
Step 2: For path ab put $s=j\omega$ & where ω varies from 0 to $+\infty$) and sketch polar plot.

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega T+1)}$$

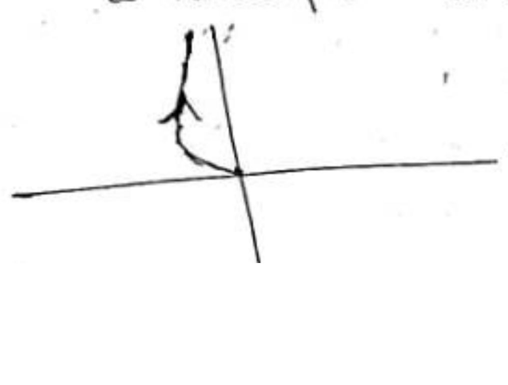
$$M = |G(j\omega)H(j\omega)| = \frac{K}{\omega\sqrt{(\omega T)^2+1}}$$

$$\phi = \angle G(j\omega)H(j\omega) = -90^\circ - \tan^{-1}(\omega T)$$

$\omega=0$	$M=\infty$	$\phi=-90^\circ$
$\omega=\infty$	$M=0$	$\phi=-180^\circ$



Step 3: For path de, it is the mirror image of path ab where ω varies from $-\infty$ to 0.



Step4: For path 'bcd' put $s = Re^{j\theta}$ where $R \to \infty$ and θ varies from $+90°$ through $0°$ to $-90°$.

$$G(j\omega)H(j\omega)\Big|_{\substack{s=Re^{j\theta} \\ R\to\infty}} = \lim_{R\to\infty} \frac{K}{Re^{j\theta}(TRe^{j\theta}+1)}$$

$$= \lim_{R\to\infty} \frac{K}{TR^2 e^{j2\theta}} = 0e^{-j2\theta}$$

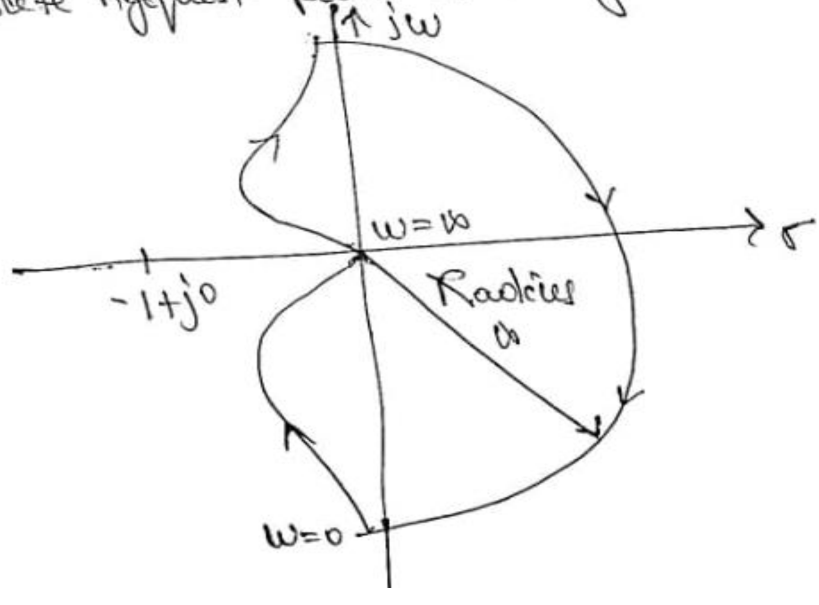
$\therefore$ $G(s)H(s)$ locus turns at the origin with zero radius from $-180°$ through $0°$ to $+180°$.

Step5: For path 'efa' put $s = \epsilon e^{j\theta}$ where $\epsilon \to 0$ and θ varies from $-90°$ through $0°$ to $+90°$.

$$G(j\omega)H(j\omega)\Big|_{\substack{s=\epsilon e^{j\theta} \\ \epsilon\to 0}} = \lim_{\epsilon\to 0} \frac{K}{\epsilon e^{j\theta}(T\epsilon e^{j\theta}+1)}$$

$$= \lim_{\epsilon\to 0} \frac{K}{\epsilon e^{j\theta}} = \infty e^{-j\theta}$$

Thus $G(s)H(s)$ locus turns at the origin with ∞ radius from $+90°$ through $0°$ to $-90°$. Hence the complete nyquist path is as given below:



Step-6: Check for stability:

As P= 0; N=0; Hence Z=0, System is stable.